1948

Stress distribution in layered systems and its applications to highway pavement design

Chung-ni Tsien
Iowa State College

STRESS DISTRIBUTION IN LAYERED SYSTEMS AND ITS APPLICATIONS TO THE HIGHWAY PAVEMENT DESIGN

by

Tsien, Chung-ni.

A Thesis Submitted to the Graduate Faculty
for the Degree of

DOCTOR OF PHILOSOPHY

Major Subject: Highway Engineering

Approved by:

Signature was redacted for privacy.

In Charge of Major Work

Signature was redacted for privacy.

Head of Major Department

Signature was redacted for privacy.

Dean of Graduate College

Iowa State College

1948

TABLE OF CONTENTS

LIST OF FIGURES

INTRODUCTION

In the field of bridges and buildings, the usual method of design is first to analyze the stresses involved by a reasonably accurate mathematical procedure and then to check these stresses with the strength of the materials used. In highway pavement design, due to the complexity of the problem, no standard procedure has been developed. Designers usually have to be satisfied with empirical methods which, experience and field tests have indicated, are workable. To place the design of highway pavements on a more rational basis, rigorous mathematical study of the stress distribution in a layered system should be invaluable.

Many investigations have been made to determine the stresses in concrete pavements under various load and pavement conditions. The most important contribution was made by Westergaard (1). His stress formulas have been verified by laboratory tests and are generally accepted in the so-called "structural design" of concrete pavements. The assumption that the subgrade pressure is proportional to the deflection is used in his analysis and it is also used in almost all similar analyses of stresses in concrete pavements. Westergaard further assumes that the subgrade is frictionless so that no shearing stress exists at the interface between the pavement and the subgrade. The author is not entirely satisfied with these assumptions for the reasons to be explained later.

In flexible pavement design, knowledge of the stresses involved and of the thickness required is not as advanced as in concrete pavement design. Mathematical analyses of the subgrade reaction under different kinds of flexible pavement have been attempted by only a few investigators. The exact performance of a flexible pavement under load is still a matter of speculation to many designers. In the author's opinion, the stress analysis for concrete pavements and that for flexible pavements is essentially the same if it is based on the theory of elasticity. The only difference is that different criteria should be used in the design of concrete pavements than in the design of flexible pavements.

The stress analysis of a layered system based on the theory of elasticity has been attempted by many investigators. Burmister (2) first suggests using the load-settlement relation obtained from the stress analysis for the design of both concrete and flexible pavements. Unlike many other authors, he treated the pavement not as a thin plate but as a layer of finite thickness. Perfect continuity on the interface was assumed. The elastic property of the subgrade was not measured by Westergaard's "modulus of subgrade reaction" but by the modulus of elasticity. Numerical solutions of the maximum deflection of a pavement under the wheel load were obtained. Burmister's method of pavement design was based on the theoretical deflection of the pavement.

In general, the method of analysis used in this study is similar in many respects to Burmister's analysis. However, the author believes that the value of Poisson's ratio used in Burmister's study should be modified using values which are closer to the values established by tests of pavement materials. Also the theoretical value of the deflection which is an infinite integral of the vertical strain may induce appreciable error due to the difference between the assumed and the actual conditions.

This study will give a more general solution of the stresses in the pavements. Bearing stresses are used as the criterion for flexible pavement design; and flexural stresses in the pavement as the criterion for concrete pavement design. Bessel-Fourier integrals are used to express the distribution of the external loading. Stresses in the radial and circumferential directions due to the external load distributed over an area can thus be evaluated. Numerical solutions of this study are made on the basis that the Poisson's ratio of the materials is equal to 0.2 instead of 0.5 (value for incompressible material) used by Burmister.

Practical applications of the method developed in this study are made in the design of flexible and concrete pavements using illustrative examples. Comparisons of the results obtained in these examples are made with the results obtained using other design formulas. In one of the examples, the method is extended to provide for the use of the results in the design of a multi-layer flexible pavement.

Since this method of analysis is based on the theory of elasticity and covers a wide range of conditions, the user of the method is expected to understand the differences between the ideal conditions assumed in the formulas and the actual conditions in a given job and their effect on the results. Laboratory verification of the results covering a wide range of conditions will, of course, be invaluable, but a laboratory investigation of the method developed herein is beyond the scope of this study.

REVIEW OF PREVIOUS STUDIES

Formulas for Design of Flexible Pavements

Ten or more design formulas for flexible pavements have been developed by various investigators. They may be grouped into the following classes:

Formulas involving arbitrary assumptions in regard to the distribution of the wheel load without considering the strength properties of the pavement

This group includes all those classical formulas which are still popularly used. The characteristic point of this group is that in each formula an arbitrary assumption is made concerning the manner of distribution of the wheel load as illustrated by Figure 1. As a result of the assumption, the formulas have the general form of:

$$t = \sqrt{\frac{W}{p}} - c$$

in which t is the required thickness; W is the total wheel load; p is the allowable subgrade pressure; and a and c are constants.

The most important formulas in this group are: Massachusetts formula (3), Downs formula (See Sheets (4)), Asphalt Institute formula (5), Harger and Bonney formula (6), Goldbeck formula (7), and Lelievre and Pons formula (8).

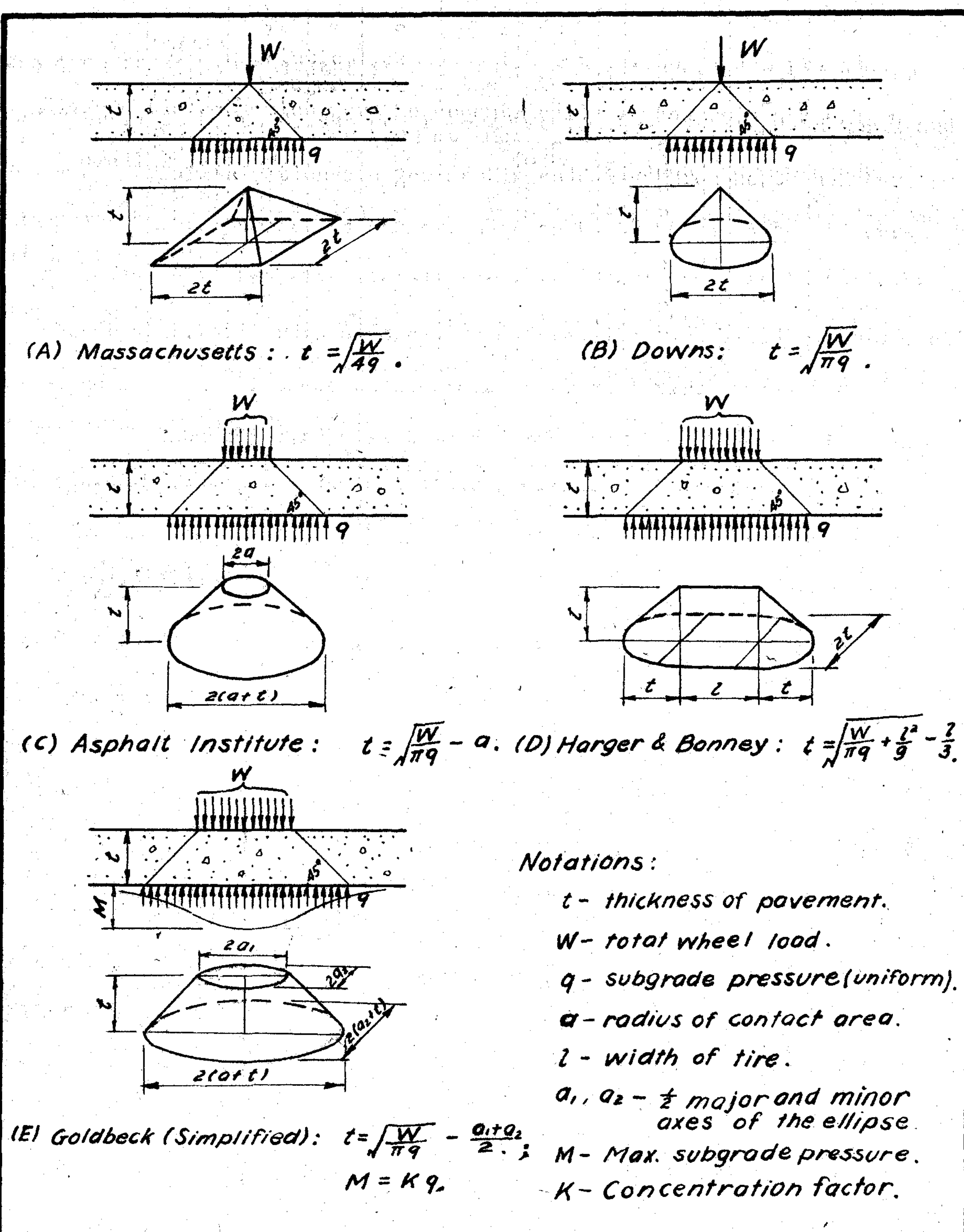


Fig.1 Design Formulas for Flexible Pavements And Assumptions made in the Distribution of the Wheel Load.

The actual maximum subgrade pressures as measured in field and laboratory tests are much higher than the pressure calculated from these formulas. A concentration factor k which is defined as the ratio between the maximum subgrade pressure and the calculated equivalent uniform pressure is suggested by Goldbeck (7). The value of k recommended by Goldbeck varies from 1.1 to 3.8 depending upon the material of the pavement. With such a large variation in the concentration factor k, it is evident that large errors are introduced in the solutions where this factor is disregarded.

Formulas involving the use of the moduli for the pavement and the subgrade

The formulas in this group are those most recently developed and are not yet widely used. This group makes use of the term "subgrade modulus" or similar term in the design formula. The proper value of the subgrade modulus of the soil in these formulas is to be determined by laboratory test.

Palmer and Barber (9) made use of triaxial compression test in determining the strength moduli of the pavement and the subgrade and then by taking into consideration the permissible deflection of the subgrade soil developed the formula:

$$t = a \sqrt[3]{\frac{C_s}{C_p}} \sqrt{\frac{p^2}{q^2} - 1}$$

in which t is the required thickness of the pavement:

a is the radius of the loaded area;

C_p and C_s are the stress-strain moduli of the pavement and the subgrade;

p is the contact pressure of the tire;

q is the allowable bearing pressure of the subgrade which is determined by the formula:

$q = C_s d_s / 1.5a$, in which d_s is the permissible deflection of the subgrade soil, recommended value of 0.1 inch.

Klinger (10) developed the following formula for flexible pavement design based on the experimental work of Hubbard and Field:

$$t = K_2 \sqrt{\frac{p}{q} - 1}$$

in which K_2 is a constant depending upon the type of pavement.

Spangler (11) developed the following formula for the design of flexible pavements based on extensive test of the load distribution through flexible surfaces of various thicknesses:

$$d_p = \frac{0.9W}{C_s} \sqrt{\frac{F}{t}}$$

in which d_p is the safe allowable deflection of the pavement in inches;

W, t, C_s have same meaning as in previous formulas;

F is the subgrade stress factor which is determined by the formula:

$$F = \frac{0.0070 + .0000068\, C_s}{\sqrt[3]{W/1000}}$$

This group of formulas shows definite improvement when compared with the previous group. However, the actual distribution of the wheel load through the pavement is still obscure. Furthermore, it should be observed that the constants in Klinger's and Spangler's formulas are dimensional, therefore the application of these formulas must be limited to the extent of the testing conditions in the tests by these men.

Formulas involving the use of the strength values for the pavement material

This group of formulas is based on the most advanced ideas in flexible pavement design and at the same time some of these formulas are the most controversial formulas of all the flexible pavement formulas. Each of these formulas has its own theory. The marked improvement of this group over the previous discussed groups is that consideration is given to the strength of the pavement in the design.

Housel (12) developed the following formula:

$$t = \frac{(p - 4m_2)b}{4m_1} + \frac{m_2 b}{p}$$

in which p is the inflation pressure of the tire, in psi.

b is the diameter of the contact area,

m_1 and m_2 are the shearing resistances of the pavement and the subgrade respectively.

Vokac (13) developed the formula based on the geometry of the pressure bulb:

$$t_b = \frac{b(p - q)}{4\sqrt{pq}};$$

$$t_s = \frac{b(p - P_s)}{4\, m_s\, R}$$

in which t_b and t_s are the thickness of the base course and the thickness of the surface course respectively,

m_s is the shearing strength of the surface course in psi,

b, p, q have the same meaning as in previous formulas,

$P_s = \frac{(p + q)^2}{4p}$ is the required minimum bearing strength of the base course, and

$$R = \frac{1}{1 - \sqrt{p/q}}$$

Hawthorn (14) used the assumption of the distribution of the wheel load some what like the combination of that in Massachusetts' formula and that in Asphalt Institute Formula, but thought that the allowable bearing value of the subgrade should be a function of the thickness, densities, and some other strength properties of the subgrade properties instead of being an independent value. His formula is:

$$q = \frac{W}{(t \tan\theta + a)^2}$$

in which θ is the angle of load distribution, a, W, t, q are same as in previous formulas. q is a function of the density of the pavement and the density of the subgrade, the thickness of the pavement, the cohesion and the coefficient of the internal friction of the subgrade soil.

Methods using charts and design curves

Since none of the existing design formulas can be used satisfactorily for all types of pavement and subgrade conditions, some highway engineers have developed charts and design curves using empirical methods as the basis of design. These methods have generally been established through long experience and are based on the satisfactory performance of existing pavements.

The most important and most widely used method in this group is the California Bearing Method. This method has been adopted with modifications by the U.S. Engineer Department (15). The bearing capacity of the subgrade is indicated by the "California Bearing Ratio" determined by laboratory test. A set of design curves based upon past experience has been developed so that for a given wheel load and for a given value of the California Bearing Ratio, the required thickness of the flexible pavement can be determined.

Besides the California Bearing Ratio method, Civil Aeronautic Administration (16) has prepared a series of design charts in which are given the required thickness of the sub-base, the base and the surface course for different plan loads and the different classification of the subgrade soil and of material used in each layer. Boyd (South Dakota Cone method) (17), Hubbard (18) and Field also have developed their respective sets of design curves based upon their own laboratory test results and on their experience.

Some results of the theoretical analysis have been plotted into charts to facilitate their use in pavement design, as for example the charts developed by Glossop and Golder (19), Smith (20), and Burmister (2).

Formulas for the Design of Concrete Pavements

Older's formula

One of the first theories for determining the thickness of a concrete pavement was developed by Older (21). By neglecting the subgrade support at the corner of the pavement and considering the wheel load as a point load, he derived the well known Older's formula:

$$t = \sqrt{\frac{kW}{f}}$$

in which t is the required thickness, W is the wheel load including the effect of impact, f is the allowable flexural stress of the concrete, and k is a constant depending upon the position of the load and the efficiency of the joints. For the free corner condition, k is equal to 3; for the edge thickness of concrete pavement with a 100 per cent efficient doweled joint, k is equal to 1.5; and for the interior thickness, assuming that the load is distributed over 4 corners, k is equal to 0.75.

A slight modification of Older's formula made to take care of the effect of subgrade support and also to give results which closely approximate computations made by Prof. Westergaard was presented by Portland Cement Association (22, p. 13).

$$t = \sqrt{\frac{kWc}{f}}$$

in which c is the subgrade bearing coefficient depending upon the bearing value of the subgrade. It varies from 0.77 to 1.10 using the smaller value for the best subgrade support and the larger value for the poorest subgrade. The recommended value of k is 1.92 for protected corners and 2.4 for unprotected corners.

Older's formula is still the most widely used formula for rigid pavement design due to its simplicity. However, since the formula is based on the theory of corner failure, it is not very logical to extend its use in determining the thickness of the interior portion of concrete pavements.

Westergaard's study and other related studies

The best known stress analysis of concrete pavement slabs is the mathematical analysis developed by Westergaard (1). His analysis involves the following assumptions:

1. The concrete slab is homogeneous, isotropic, elastic, and uniform in thickness.

2. The reaction of the subgrade is vertical only and its intensity is proportional to the deflection of the slab. The ratio between the intensity of the subgrade reaction and the deflection of the slab is defined as the modulus of subgrade reaction.

For three positions of load, the analysis gives the following results:

$$f_c = \frac{3W}{t^2}\left[1 - \left(\frac{12(1-\mu^2)K}{E\,t^3}\right)^{0.15}(a\sqrt{2})^{0.6}\right]$$

$$f_i = 0.275\ (1+\mu)\ \frac{W}{t^2}\ \log_{10}\left(\frac{Et^3}{Kb^4}\right)$$

$$f_e = 0.529(1+0.54\mu)\frac{W}{t^2}\left[\log_{10}\left(\frac{Et^3}{Kb^4}\right) - 0.71\right]$$

in which

W is the total wheel load in pounds;

f_c, f_i, and f_e are the maximum flexural stresses in pounds per square inch in the slab due to the load W at the corner, at an interior point, and at the edge of the pavement respectively;

t is the thickness of the slab in inches;

μ is the Poisson's ration for concrete;

E is the modulus of elasticity of the concrete in pounds per square inch;

K is the subgrade modulus in pounds per square inch per inch;

a is the radius of the contact area of the wheel load on the pavement in inches;

b is the equivalent radius of the resisting sections supporting the pavement in inches,

$b = \sqrt{1.6a^2 + t^2} - 0.675t$ when $a < 1.72t$

$b = a$ when $a > 1.72t$.

Modifications of these formulas based on a reduction of the subgrade support and of the maximum deflection have also been made. These modifications, in the author's opinion, are empirical and will not be discussed here.

It is obvious that the validity of the above formulas by Westergaard depends upon the validity of his assumptions. The idea of the "subgrade modulus" involves: first, the proportionality of the distribution of the subgrade reaction and of the deflection of the surface of the subgrade and second, the proportional ratio must be a constant independent of the type of loading. The first part of the assumption is approximately correct for the usual pavement slab subgrade where the rate of change of the slope of the deflected surface is low. However, this is no longer true when the rigid plate method is used in determining the modulus. It can be mathematically proved (23, p. 339) and has been demonstrated in tests (22, p. 55) that the value of subgrade modulus so determined is materially effected by the size and type of loaded area. Thus far a rational method has not been developed to determine the "proportional constant" which is by no means a constant, so that it will fit correctly for use in the Westergaard formulas. It should also be noted here that some authors (22, 24) have stated that the error in the stresses determined by these formulas due to variations in the subgrade modulus is negligible. This seems to imply that including the "subgrade modulus" in these formulas is unnecessary which is incompatible with the original purpose of the analysis.

Many investigators have made studies of the stresses in concrete pavement slabs especially at the free corners. Formulas based on laboratory and field tests were developed by Spangler (25), Bradbury (26), Kelley (24), and Portland Cement Association (22). They are in general of the following form and may be called empirical modifications of Westergaard's results:

$$\frac{f_c t^2}{3W} = m\left[1 - \left(\frac{a_1}{R}\right)^n\right]$$

in which

f_c, t, and W have the same meaning as in Westergaard's formulas;

$a_1 = a\sqrt{2}$;

$R = \left(\frac{Et^3}{12(1 - \ ^2)K}\right)^{\frac{1}{4}}$, is called the radius of relative stiffness, in inches;

m and n are constants which vary from 0.6 to 1.2.

Related Problems Using Theory of Elasticity

Boussinesq's solution of the stress distribution in homogeneous elastic semi-infinite bodies

In 1885, Boussinesq (27) also see Timoshenko (23), published his famous solution of the stress distribution in homogeneous elastic semi-infinite bodies. His theory has recently found wide application in soil mechanics for evaluating the stresses or pressure in soil under load. For a point load W applied on the surface at the origin, the stresses in cylindrical coordinate system (r,e,z) are:

$$S_{zz} = \frac{3W}{2\pi z^2} \cos^5\phi$$

$$S_{rr} = \frac{W}{2\pi z^2}\left(3\cos^3\phi\sin^2\phi - (1-2\mu)\frac{\cos^2\phi}{1+\cos\phi}\right)$$

$$S_{ee} = -(1-2\mu)\frac{W}{2\pi z^2}\left[\cos^3\phi - \frac{\cos^2\phi}{1+\cos\phi}\right]$$

$$S_{rz} = \frac{3W}{2\pi z^2} \cos^4\phi\sin\phi$$

in which μ is the Poisson's ratio of the material and $\phi = \tan^{-1}\frac{r}{z}$.

The computation of stresses due to loads uniformly distributed over a rectangular or a circular area is rather involved, and the results cannot be expressed readily in a set of simple equations. However, the problem has been solved and the results have been compiled in tables which make it possible to determine the stresses at any point by means of these tables (See Love 28).

At certain special points, some of the stresses due to a circular uniformly distributed load of intensity p and radius a can be reduced to simple forms. The following formulas are of special interest to highway engineers:

(1) The formula for the deflection of the surface at the center of the loaded area is:

$$d = \frac{2(1 - \mu^2)ap}{E}$$

(2) The formula for vertical normal stress at a depth z below the center of the loaded area is: $S_{zz} = p(1 - \cos^3\theta)$

where $\theta = \tan^{-1}\frac{a}{z}$

(3) The formula for the horizontal normal stress at the same point is:

$$S_{rr} = \frac{p}{2}\left[-1 - 2\mu + 2(1 + \mu)\cos\theta - \cos^3\theta\right]$$

(4) The formula for the settlement of a rigid plate under a total load W bearing on an infinite elastic body is:

$$d = \frac{W(1 - \mu^2)}{2aE}$$

Studies of problems of layered systems

The studies made on the stress-distribution in layered systems may be divided into three classes:

The first class includes those studies made on an elastic layer on a rigid base. The base may be assumed to be frictionless or it may be assumed to be cohesive. Melan (29) solved the problem for a point load and for a line load acting on an elastic layer resting on a frictionless rigid base. Marguerre (30) solved the problem for a line load on an elastic layer resting on a rough rigid base and Passer (31) and Biot (32) solved that problem for a point load. All the formulas derived are very lengthy and the results do not have any close relation with this study.

The second class includes the studies made on a thin plate rested on an elastic base. Unsatisfied with Westergaard's assumption and use of the "modulus of subgrade reaction", Hogg (33) made an attempt to solve for the flexural stress in a concrete pavement. He dealt with the problem of an infinite plate resting on a frictionless elastic base. Murphy (34) made the study for finite rectangular plates. He used the modulus of subgrade reaction as the measure of the elastic property of the subgrade as Westergaard did.

The third class includes the problem of an elastic layer of finite thickness resting on a semi-infinite elastic base. Marguerre (35), making the use of Bessel functions and Fourier integrals, first obtained the general solution of the axially symmetric stress distribution in a "thick plate". To simplify the problem, he assumed that the ratio between the Lame's constants of the thick plate and that of the base are equal. He further assumed that the base is frictionless. Examples were given in which the deflections and the reaction of the base for a point load acting on the surface of the plate. Burmister (2) published his work on the study of anelastic layer resting on a rough elastic base. He simplifies the analysis by assuming the materials are incompressible (that is to say, that Poisson's ratio is equal to 0.5). A solution for the deflection of the surface at the center of a circular uniformly loaded area was derived and successfully evaluated by replacing the integrand of the formula with a series expansion and integrating the series term by term.

STRESS DISTRIBUTION IN LAYERED SYSTEMS

Theoretical Solution

<u>Notations</u> (See Figure 2)

E - Modulus of elasticity.

μ - Poisson's ratio.

r, θ, z - Cylindrical coordinates.

S - Stress.

S_{rr}, $S_{\theta\theta}$, S_{zz} - Normal stress components in cylindrical coordinates.

$S_{r\theta}$, $S_{\theta z}$, S_{zr} - Shearing stress components in cylindrical coordinates.

ϕ - Stress function.

u, w - Displacement components in the direction of r and z respectively.

t - Thickness of the top layer.

r - Radius of the circular loaded area.

p - Intensity of the uniformly distributed load.

$a = r/t$ Reciprocal of thickness ratio.

E_1 = Modulus of elasticity of the top layer.

E_2 = Modulus of elasticity of the base.

k - E_1/E_2, Stiffness ratio.

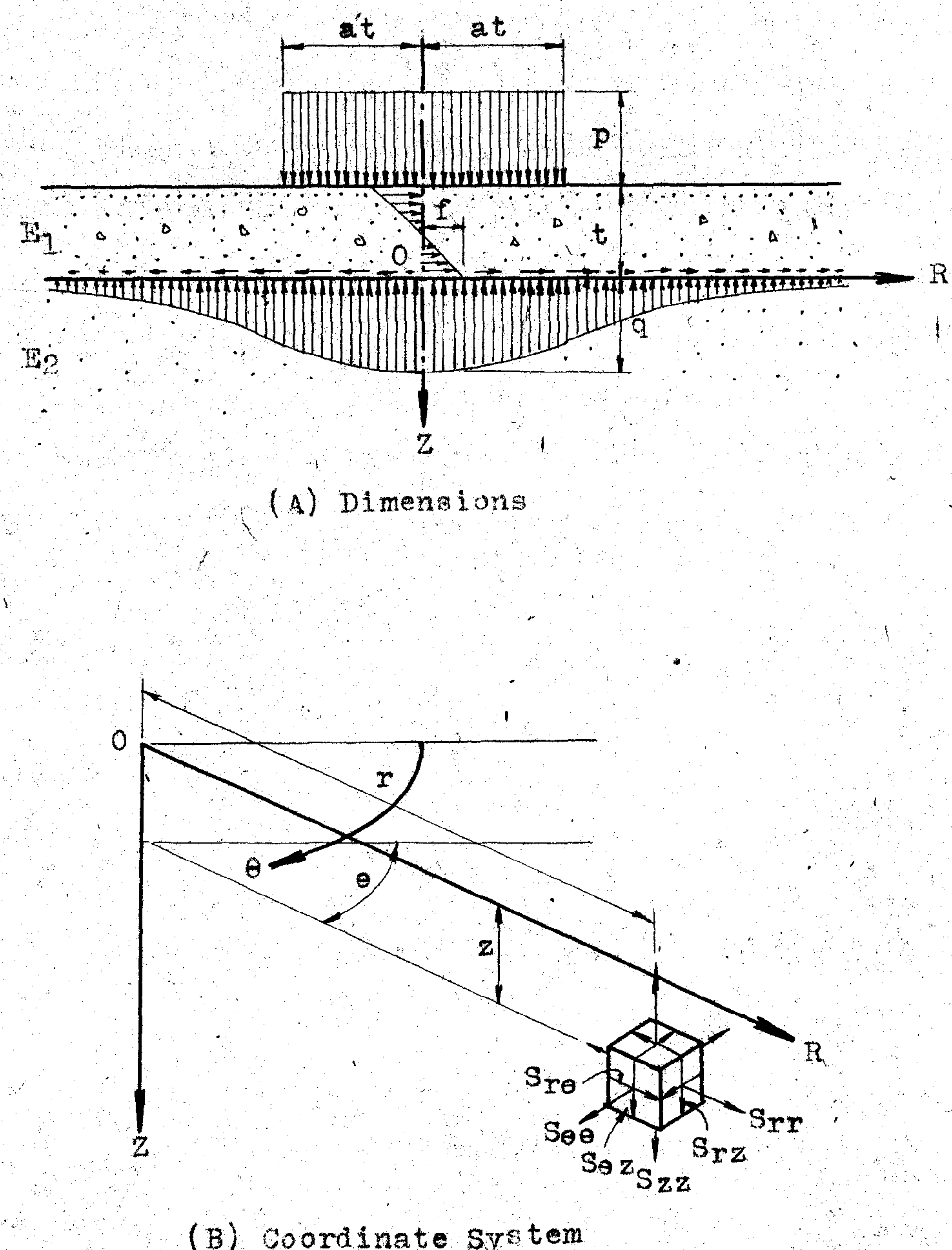


Fig.2 Dimensions, Notations and Coordinate System used in this Study

$n = E_2/E_1$ Reciprocal of stiffness ratio.

m - Parameter.

Assumptions

a. The materials in this study are assumed to be homogeneous, isotropic, and elastic. The displacements are so small that the linear theory of elasticity is considered to be valid for these materials.

b. Poisson's ratio is the same for all materials in a given problem.

c. The weight of the materials is neglected, that is, only the portion of stress which results from the external load is considered.

d. The external load is axially symmetric, and normal to the boundary.

e. The displacement is continuous across the intersurface.

f. Every point is in static equilibrium.

The justification of these assumptions will be discussed in the applications to highway pavement design (Chapter IV).

General solution for axially symmetric stress distribution

For three dimensional problems with axial symmetry, the following relationships have been developed [See Timoshenko (23) and Burmister (2)]:

a. Equations of equilibrium.

$$\frac{\partial S_{rr}}{\partial r} + \frac{\partial S_{rz}}{\partial z} + \frac{S_{rr} - S_{\theta\theta}}{r} = 0 \qquad (1)$$

$$\frac{\partial S_{rz}}{\partial r} + \frac{\partial S_{zz}}{\partial z} + \frac{S_{rz}}{r} = 0$$

$$S_{r\theta} = 0$$
$$S_{\theta z} = 0$$

b. The stress function and the stresses in equations (1) are satisfied if:

$$S_{rr} = \frac{\partial}{\partial z}\left[\mu\nabla^2\phi - \frac{\partial^2 \phi}{\partial r^2}\right]$$

$$S_{\theta\theta} = \frac{\partial}{\partial z}\left[\mu\nabla^2\phi - \frac{1}{r}\frac{\partial\phi}{\partial r}\right]$$

$$S_{zz} = \frac{\partial}{\partial z}\left[(2 - \mu)\nabla^2\phi - \frac{\partial^2\phi}{\partial z^2}\right]$$

$$S_{rz} = \frac{\partial}{\partial r}\left[(1 - \mu)\nabla^2\phi - \frac{\partial^2\phi}{\partial z^2}\right]$$

where $\nabla^2 = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right)$ (2)

c. The compatibility equation is: (3)

$$\nabla^4 \phi = 0.$$

d. The displacement is:

$$u = \frac{-(1 + \mu)}{E}\frac{\partial^2\phi}{\partial r\partial z}$$

$$w = \frac{(1 + \mu)}{E}\left[2(1 - \mu)\nabla^2\phi - \frac{\partial^2\phi}{\partial z^2}\right] \quad (4)$$

e. Solution. The solution of the compatibility equation (3) is:

$$\phi = J_0(mr)\ (Ae^{mz} + Be^{-mz} + Cze^{mz} + Dze^{-mz})\ ;$$

in which J_0 is the Bessel function of the first kind of order 0, and A, B, C, D are arbitrary constants. To find the stresses, substitute the result of the compatibility equation into the stress equations (2) to obtain:

$$S_{zz} = -m^2 J_o(mr) \left[Ame^{mz} - Bme^{-mz} + C(mz - 1 + 2\mu) e^{mz} + D(-mz - 1 + 2\mu)e^{-mz} \right]$$

$$S_{rr} = m^2 J_o(mr) \left[Ame^{mz} - Bme^{-mz} + C(mz + 1 + 2\mu)e^{mz} + D(-mz + 1 + 2\mu)e^{-mz} \right] - \frac{mJ_1(mr)}{r} \left[Ame^{mz} - Bme^{-mz} + C(mz + 1)e^{mz} - D(-mz + 1)e^{-mz} \right]$$

$$S_{\theta\theta} = 2\mu m^2 J_o(mr) \left(Ce^{mz} + De^{-mz} \right) + \frac{mJ_1(mr)}{r} \left[Ame^{mz} - Bme^{-mz} + C(mz + 1)e^{mz} + D(-mz + 1)e^{-mz} \right]$$

$$S_{rz} = m^2 J_1(mr) \left[Ame^{mz} + Bme^{-mz} + C(mz + 2\mu)e^{mz} + D(mz - 2\mu)e^{-mz} \right] \qquad (5)$$

The displacement components are:

$$w = \frac{-(1 + \mu)}{E} mJ_o(mr) \left[Ame^{mz} + Bme^{-mz} + C(mz - 2 + 4\mu)e^{mz} + D(mz + 2 - 4\mu)e^{-mz} \right]$$

$$u = \frac{(1 + \mu)}{E} mJ_1(mr) \left[Ame^{mz} - Bme^{-mz} + C(mz + 1)e^{mz} + D(-mz + 1)e^{-mz} \right] \qquad (5a)$$

<u>Solution for axially symmetrical stress distribution in layered systems</u>

The boundary conditions to be satisfied in a layered system are: (a) the continuity of the displacement components u and w across the intersurface, (b) the equilibrium of the points on the intersurface, (c) the equilibrium with respect to the external wall, and (d) the vanishing of the stresses at infinity. All factors of the top layer are denoted by the subscript 1 and those of the base by a subscript 2 as in Figure 2. If the external load is distributed as F(r) the boundary conditions and equations are:

When $z = -t$: $(Szz)_1 = -F(r)$, $(Srz)_1 = 0$;

When $z = 0$, $(Szz)_1 = (Szz)_2$

$(Szr)_1 = (Szr)_2$,

$(w)_1 = (w)_2$,

$(u)_1 = (u)_2$;

and when $z = \infty$,

$$(\phi)_2 = 0. \qquad (6)$$

The loading function F(r) can be expressed by a Fourier-Bessel Integral. [See Whittaker (37).]

$$F(r) = \int_0^\infty mJo\,(mr) \int_0^\infty xF(x)\; J_0(mx)\; dx\; dm.$$

For a circular uniformly distributed load of radius $r = at$, and of intensity p:

$F(r) = p$, when $r < at$;

$F(r) = 0$, when $r > at$.

$$\int_0^{at} Jo(mx)\; x\; dx = \frac{at}{m} J_1(mat)\quad.$$

$$F(r) = atp \int_0^\infty J_0(mr) J_1(mat)\; dm.$$

Now, let Q_m be defined as:

$$Q_m \left[f(r,e,z,m) \right] = atp \int_0^\infty f(r,e,z,m) J_1(mat)\; dm \qquad (7).$$

Q_m is a mathematical operation on the function $f(r,e,z,m)$ with respect to the parameter m. The order of operation of Q_m and the integration or differentiation with respect to r, o and z in any formula can be interchanged.

The values of the stresses given by equations (5, 5a) are substituted into the boundary conditions (6) and these equations are simplified to obtain:

$$A_1 me^{-mt} - B_1 me^{mt} + C_1(-mt - 1 + 2\mu)e^{-mt} + D_1(mt - 1 + 2\mu)e^{mt} = -Q_m/m;$$

$$A_1 me^{-mt} + B_1 me^{mt} + C_1(2\mu - mt)e^{-mt} + D_1(-2\mu - mt)e^{mt} = 0;$$

$$(A_1 - B_1 + B_2)m - (1 - 2\mu)(C_1 + D_1 - D_2) = 0;$$

$$(A_1 + B_1 - B_2)m + 2\mu(C_1 - D_1 + D_2) = 0;$$

$$\left(\frac{A_1}{E_1} + \frac{B_1}{E_1} - \frac{B_2}{E_2}\right)m - 2(1 - 2\mu)\left(\frac{C_1}{E_1} - \frac{D_1}{E_1} + \frac{D_2}{E_2}\right) = 0;$$

$$\left(\frac{A_1}{E_1} - \frac{B_1}{E_1} + \frac{B_2}{E_2}\right)m + \frac{C_1}{E_1} + \frac{D_1}{E_1} - \frac{D_2}{E_2} = 0;$$

$$A_2 = 0;$$

$$C_2 = 0. \qquad (8)$$

Solving for the six constants:

$$A_1 = \frac{1}{m^3} Q_m \left[\frac{1}{N} (1 - n) \left\{ \left[mt(4\mu - 1)(n + 3 - 4\mu) + (2\mu - 1)(4\mu - 1)n + (2\mu + n)(3 - 4\mu) \right] e^{mt} - (1 - n)(3 - 4\mu)(2\mu - mt)e^{-mt} \right\} \right]$$

$$B_1 = \frac{1}{m^3} Q_m \left[\frac{1}{N} \left\{ (mt + 2\mu)(3 - 4\mu + n)(3n - 4\mu n + 1)e^{mt} + (1 - n) \left[mt(4\mu - 1)(n + 3 - 4\mu) - (2\mu - 1)(4\mu - 1)n - (2\mu + n)(3 - 4\mu) \right] e^{-mt} \right\} \right]$$

$$C_1 = \frac{1}{m^2} Q_m \left[\frac{1}{N}(1 - n) \left\{ -(3 - 4\mu + n)(1 + 2\, mt)e^{mt} + (1 - n)(3 - 4\mu)e^{-mt} \right\} \right]$$

$$D_1 = \frac{1}{m^2} Q_m \left[\frac{1}{N} \left\{ (3 - 4\mu - n)\ (3n - 4\mu n + 1)e^{mt} + (1 - n)(2\, mt - 1)e^{-mt} \right\} \right]$$

$$B_2 = \frac{1}{m^2} 4n(1 - \mu)Q_m \left[\frac{1}{N} \left\{ \left[mt(3 - 4\mu + n) + 1 - n + 8n\mu(1 - \mu) \right] e^{mt} + (1 - n) \left[mt(4\mu - 1) - 1 \right] e^{-mt} \right\} \right]$$

$$D_2 = \frac{1}{m^2} 4n(1 - \mu)Q_m \left[\frac{1}{N} \left\{ (3n - 4n\mu + 1)e^{mt} + (1 - n)(2\, mt - 1)e^{-mt} \right\} \right]$$

in which:

$$n = E_2/E_1$$

$$N = - \left\{ (1 - n)^2(3 - 4\mu)e^{2mt} + (3n - 4\mu n + 1)(3 - 4\mu + n)e^{2mt} + (n - 1) \left[(3 - 4\mu + n)(1 + 4m^2t^2) + (3 - 4\mu)(3n - 4\mu n + 1) \right] \right\}.$$

Substituting these constants into equations (8, 8a):

$$(S_{zz})_1 = -Q_m \left[\frac{Jo(mr)}{N} \left\{ (1 - n) \left[mt(3 + 2\mu - n)(1 - 2mz) + 2(2\mu - 1)^2 n + (1 + n)(3 - 4\mu) - mz(3 - 4\mu + n) \right] e^{m(t+z)} + (1 - n)^2\ (mt + mz - 1)(3 - 4\mu)\ ^{m(z-t)} + (3 - 4\mu + n)(3n - 2\ \ n + 1)(-mt - mz - 1)e^{m(t-z)} + (1 - n) \left[mt(3 - 4\mu + n)(-1 - 2mz) + 2(2\mu - 1)^2 n + (1 + n)\ (3 - 4\mu) + mz(3 - 4\mu + n) \right] e^{-m(t+z)} \right\} \right]$$

$$(S_{rz})_1 = Q_m \left[\frac{J_1(mr)}{N} \left\{ (1-n) \left[mt(3-4\mu + n)(-1-2mz) - (3-4\mu + n)mz + 4\ (1-\mu)(1 - 2\mu)n \right] e^{m(t+z)} + (1-n)^2(3-4\mu)m(t+z)e^{m(z-t)} + (3 - 4\mu + n)(3n - 4\mu n + 1)m(t + z)\ \ ^{m(t-z)} + (1 - n) \left[mt(3 - 4\mu + n)(2\, mz - 1) - (3 - 4\mu + n)mz - 4\ (1 - \mu)(1 - 2\mu)n \right] e^{-m(t+z)} \right\} \right]$$

$$(S_{rr})_1 = Q_m \Bigg[\frac{J_0(mr)}{N} \Big\{ (1 - n) \big[mt(3-4\mu + n)(-3-2mz)$$

$$- (mz + 1)(3 - 4\mu + n) + 4 (1-\mu)(1 - 2\mu)n \big] e^{m(z+t)}$$

$$+ (1-n)^2 (3 - 4\mu)(mz + mt + 1) e^{m(z-t)}$$

$$+ (3 - 4\mu + n)(3n - 4\mu n + 1)(1 - mz - mt) e^{m(t-z)}$$

$$+ (1 - n) \big[mt (3 - 4\mu + n)(3 - 2mz) + (mz - 1)(3 - 4\mu + n)$$

$$+ 4(1 - \mu)(1 - 2\mu)n \big] e^{-m(t+z)} \Big\} - \frac{J_1(mr)}{mrN} \Big\{ (1-n)$$

$$\big[mt (3 - 4\mu + n)(4\mu - 3 - 2mz) - mz(3 - 4\mu + n)$$

$$- (1 - 2\mu)(3 - 4\mu)(1 - n) \big] e^{m(t+z)}$$

$$+ (3 - 4\mu + n)(3n - 4\mu n + 1)(-mz - mt + 1 - 2\mu) e^{m(t-z)}$$

$$+ (1 - n)^2 (3 - 4\mu)(mz + mt + 1 - 2\mu) e^{m(z-t)}$$

$$+ (1 - n) \big[mt (3-4\mu + n)(3 - 4\mu - 2 mz) + mz (3-4\mu + n)$$

$$- (1 - 2\mu)(3 - 4\mu)(1 - n) \big] e^{-m(t+z)} \Big\} \Bigg]$$

$$(S_{\theta\theta})_1 = Q_m \Bigg[2\mu \frac{J_0(mr)}{N} \Big\{ - (1-n)(3 - 4\mu + n)(1+2 mt) e^{m(t+z)}$$

$$+ (1-n)^2 (3 - 4\mu) e^{m(z-t)} + (3 - 4\mu + n) (3n - 4\mu n + 1) e^{m(t-z)}$$

$$+ (1-n)(3-4\mu + n)(2 mt - 1) e^{-m(t+z)} \Big\}$$

$$+ \frac{J_1(mr)}{mrN} \Big\{ (1-n) \big[mt(3 - 4\mu + n)(4\mu - 3 - 2 mz) - mz(3-4\mu + n)$$

$$- (1 - 2\mu)(3 - 4\mu)(1-n) \big] e^{m(t+z)} + (1 - n)^2 (3 - 4\mu)$$

$$(mt + mz + 1 - 2\mu) e^{m(z-t)} + (3 - 4\mu + n)(3n - 4\mu n + 1)$$

$$(-mz - mt + 1 - 2\mu) e^{m(t-z)} + (1-n) \big[mt (3 - 4\mu + n)$$

$$(3 - 4\mu - 2mz) + mz(3 - 4\mu + n) - (1 - 2\mu) (3 - 4\mu)$$

$$(1 - n) \big] e^{-m(t+z)} \Big\} \Bigg]$$

$$
\begin{aligned}
(w)_1 = \frac{-(1+\mu)}{E_1} Q_m \Bigg[\frac{J_o(mr)}{mN} \Big\{ & (1-n)\big[mt(3-4\mu-2mz)(3-4\mu+n) \\
& - mz(3-4\mu+n) + 2(3-4\mu)(1-\mu)(1+n)\big] e^{m(t+z)} \\
& + (1-n)^2 (3-4\mu)(mt+mz-2+2\mu) e^{m(z-t)} \\
& + (3-4\mu+n)(3n-4\mu n+1)(mt+mz+2-2\mu) e^{m(t-z)} \\
& + (1-n)\big[mt(3-4\mu+n)(3-4\mu+2mz) - mz(3-4\mu+n) \\
& - 2(3-4\mu)(1-\mu)(1+n)\big] \; e^{-m(t+z)} \Big\} \Bigg]
\end{aligned}
$$

$$
\begin{aligned}
(u)_1 = \frac{(1+\mu)}{E_1} Q_m \Bigg[\frac{J_1(mr)}{mN} \Big\{ & (1-n)\big[mt(3-4\mu+n)(4\mu-3-2mz) \\
& - mz(3-4\mu+n) - (1-2\mu)(3-4\mu)(1-n)\big] e^{m(t+z)} \\
& + (1-n)^2 (3-4\mu)(mt+mz+1-2\mu) e^{m(z-t)} \\
& + (3-4\mu+n)(3n-4\mu n+1)(-mt-mz+1-2\mu) e^{m(t-z)} \\
& + (1-n)\big[mt(3-4\mu+n)(3-4\mu-2mz) + mz(3-4\mu+n) \\
& - (1-2\mu)(3-4\mu)(1-n)\big] e^{-m(t+z)} \Big\} \Bigg]
\end{aligned}
$$

$$
\begin{aligned}
(S_{zz})_2 = -4n(1-\mu) Q_m \Bigg[\frac{J_o(mr)}{N} \Big\{ & \big[2(\mu-1)(1+n) - mz(3n-4\mu n+1) \\
& - mt(3-4\mu+n)\big] e^{m(t-z)} + (1-n)\big[mt(-1-2mz) + 2 - 2\mu + mz\big] e^{-m(t+z)} \Big\} \Bigg]
\end{aligned}
$$

$$
\begin{aligned}
(S_{rz})_2 = 4n(1-\mu) Q_m \Bigg[\frac{J_1(mr)}{N} \Big\{ & \big[mt(3-4\mu+n) + mz(3n-4\mu n+1) \\
& + 2(1-\mu)(1-n)\big] e^{m(t-z)} + (1-n)\big[mt(2mz-1) \\
& + 2\mu - 1 - mz\big] e^{-m(z+t)} \Big\} \Bigg]
\end{aligned}
$$

$$
\begin{aligned}
(S_{rr})_2 = 4n(1-\mu) Q_m \Bigg[\frac{J_o(mr)}{N} \Big\{ & \big[-mz(3n-4\mu n+1) - mt(3-4\mu+n) \\
& + 3n(1-2\mu) + n + 2\mu\big] e^{m(t-z)} + (1-n)\big[mt(3-2mz) \\
& + mz - 2\mu\big] e^{-m(t-z)} \Big\} + \frac{J_1(mr)}{mrN} \Big\{ \big[-mt(3n-4\mu n+1) \\
& - mt(3-4\mu+n) + 4n(1-\mu)(1-2\mu)\big] e^{m(t-z)} \\
& + (1-n)\big[mt(3-4\mu-2mz) + mz\big] e^{-m(t+z)} \Big\} \Bigg]
\end{aligned}
$$

$$(S_{\theta\theta})_2 = 4n(1-\mu) \quad Q_m \left[2\mu \frac{J_o(mr)}{N} \left\{ (3n - 4\mu n + 1) e^{m(t-z)} + (1 - n)(2mt - 1) e^{-m(t+z)} \right\} + \frac{J_1(mr)}{mrN} \left\{ \left[-mt\,(3n - 4\mu n + 1) - mt(3 - 4\mu + n) + 4n(1 - \mu)(1 - 2\mu) \right] e^{m(t-z)} + (1-n)\left[mt\,(3 - 4\mu - 2mz) + mz \right] e^{-m(t+z)} \right\} \right]$$

$$(w)_2 = \frac{-2(1-\mu^2)}{E_1} \quad Q_m \left[\frac{J_o(mr)}{mN} \left\{ \left[mt\,(3 - 4\mu + n) + mz\,(3n - 4\mu n + 1) + (3 - 4\mu) + (5 - 12\mu + 8\mu^2)n \right] e^{m(t-z)} + (1 - n)\left[mt(3 - 4\mu + 2mz) - (mz + 3 - 4\mu) \right] e^{-m(t+z)} \right\} \right]$$

$$(u)_2 = \frac{4(1-\mu^2)}{E_1} \quad Q_m \left[\frac{J_1(mr)}{mN} \left\{ \left[mz\,(3n - 4\mu n + 1) - mt\,(3 - 4\mu + n) + 4n\,(1 - \mu)(1 - 2\mu) \right] e^{m(t-z)} + (1 - n)\left[mt\,(3 - 4\mu - 2mz) + mz \right] e^{-m(t+z)} \right\} \right] \qquad (9)$$

At the interface ($z = 0$):

$$(S_{zz})_1 = (S_{zz})_2 = -4n(1-\mu) \quad Q_m \left[\frac{J_o(mr)}{N} \left\{ \left[2\,(\mu - 1)(1 + n) - mt(3 - 4\mu + n) \right] e^{mt} + (1 - n)(2 - 2\mu - mt) e^{-mt} \right\} \right]$$

$$(S_{rz})_1 = (S_{rz})_2 = 4(1-\mu)n\, Q_m \left[\frac{J_1(mr)}{N} \left\{ \left[mt(3 - 4\mu + n) + (n - 1)(2\mu - 1) \right] e^{mt} + (1 - n)\,(2\mu - 1 - mt) e^{-mt} \right\} \right]$$

$$(w)_1 = (w)_2 = \frac{-4(1-\mu^2)}{E_1} \quad Q_m \left[\frac{J_o(mr)}{N} \left\{ \left[mt(3 - 4\mu + n) + 3 - 4\mu + n(5 - 12\mu + 8\mu^2) \right] e^{mt} + (1 - n)(3 - 4\mu)(mt - 1) e^{-mt} \right\} \right]$$

$$(u)_1 = (u)_2 = \frac{4(1-\mu^2)}{E_1} \quad Q_m \left[\frac{J_o(mr)}{mN} \left\{ \left[-mt(3 - 4\mu + n) + 4n(1 - \mu)(1 - 2\mu) \right] e^{mt} + (1 - n)mt(3 - 4\mu) e^{-mt} \right\} \right]$$

$$(S_{\theta\theta})_1 = 2\mu Q_m \left[\frac{J_o(mr)}{N} \left\{ 2(3 - 4\mu + n)\left[mt(n - 1) + 2n(1 - \mu) \right] e^{mt} + 2(1 - n)\left[mt(3 - 4\mu + n) - 2n(1 - \mu) \right] e^{-mt} \right\} \right] + 4(1 - \mu)\, Q_m \left[\frac{J_1(mr)}{mrN} \left\{ \left[-mt(3 - 4\mu + n) + 4n(1 - \mu)(1 - 2\mu) \right] e^{mt} + mt(1 - n)(3 - 4\mu) e^{-mt} \right\} \right]$$

$$(S\theta\theta)_2 = 4n(1-\mu)\ Q_m\left[\frac{2\mu J_o(mr)}{N}\left\{(3n-4\mu n+1)e^{mt}+(1-n)(2mt-1)e^{-mt}\right\}\right.$$
$$+\frac{J_1(mr)}{mrN}\left\{\left[-mt(3-4\mu+n)-4n(1-\mu)(1-2\mu)\right]e^{mt}\right.$$
$$\left.\left.+mt(1-n)(3-4\mu)e^{-mt}\right\}\right]$$

$$(Srr)_1 = 4Q_m\left[\frac{J_o(mr)}{N}\left\{\left[(3-4\mu+n)(\mu n-1)mt+8n(1-\mu)(\mu n+2-3\mu)\right]e^{mt}\right.\right.$$
$$\left.+(1-n)\ mt(\mu n+3-4\mu)+\mu n\ (\mu-1)e^{-mt}\right\}-\frac{(1-\mu)J_1(mr)}{mrN}$$
$$\left.\left\{\left[-mt(3-4\mu+n)+4n(1-\mu)(1-2\mu)\right]e^{mt}+(1-n)mt(3-4\mu)e^{-mt}\right\}\right]$$

$$(Srr)_2 = 4(1-\mu)n\ Q_m\left[\frac{J_o(mr)}{N}\left\{\left[3n(1-2\mu)+n-2\mu-mt(3-4\mu+n)\right]e^{mt}\right.\right.$$
$$\left.+(1-n)(3\ mt\ \cdots\ 2\mu)e^{-mt}\right\}-\frac{J_1(mr)}{mrN}\left\{\left[-mt(3-4\mu+n)\right.\right.$$
$$\left.\left.\left.+4n(1-\mu)(1-2\mu)\right]e^{mt}+(1-n)mt(3-4\mu)e^{-mt}\right\}\right] \qquad (9a)$$

On the top surface $(z = -t)$:

$$(Szz)_1 = Q_m\left[J_o(mr)\right] = F(r)$$

$$(Srz)_1 = 0$$

$$(w)_1 = \frac{-2(1-\mu^2)}{E_1}\ Q_m\left[\frac{J_o(mr)}{mN}\left\{(3-4\mu+n)(3n-4\mu n+1)e^{2mt}\right.\right.$$
$$\left.\left.-(1-n)^2(3-4\mu)e^{-2mt}+(1-n)\left[4\ mt(3-4\ \ +n)-2(1+n)(3-4\ \right]\right\}\right]$$

$$(u)_1 = \frac{(1+\mu)}{E_1}\ Q_m\left[\frac{J_1(mr)}{mN}\left\{(1-2\mu)(3-4\mu+n)(3n-4\mu n+1)e^{2mt}\right.\right.$$
$$+(1-n)^2(3-4\mu)(1-2\mu)e^{-2mt}+(1-n)\left[4m^2t^2(3-4\mu+n)\right.$$
$$\left.\left.\left.+2(2\mu-1)(3-4\mu)(1-n)\right]\right\}\right]$$

$$(Srr)_1 = Q_m\left[\frac{J_o(mr)}{N}\left\{(3-4\mu+n)(3n-4\mu n+1)e^{2mt}+(1-n)^2(3-4\mu)e^{-2mt}\right.\right.$$
$$\left.+(1-n)\left[(3-4\mu+n)(4m^2t^2-2)+8(1-\mu)(1-2\mu)n\right]\right\}$$
$$-\frac{J_1(mr)}{mrN}\left\{(1-2\mu)(3-4\mu+n)(3n-4\mu n+1)e^{2mt}\right.$$
$$+(1-n)^2(3-4\mu)(1-2\mu)e^{-2mt}+(1-n)\left[2m^2t\ (3-4\mu+n)\right.$$
$$\left.\left.\left.-2(1-2\mu)(1-n)(3-4\mu)\right]\right\}\right]$$

$$(S_{\theta\theta})_1 = 2\mu\, Q_m \Bigg[\frac{J_o(mr)}{N} \Big\{ (3 - 4\mu + n)(3n - 4\mu n + 1)e^{2mt} + (1 - n)^2(3 - 4\mu)e^{-2mt}$$
$$-2(1 - n)(3 - 4\mu + n) \Big\} + \frac{J_1(mr)}{mrN} \Big\{ (1 - 2\mu)(3 - 4\mu + n)(3n - 4\mu n + 1)e^{2mt}$$
$$(1 - n)\ (3 - 4\mu)\ (1 - 2\mu)e^{-2mt} + (1 - n) \big[4m^2t^2(3 - 4\mu + n)$$
$$-2(1 - 2\mu)(3 - 4\mu)(1 - n) \big] \Big\} \Bigg] \qquad (9b)$$

As a simple check, it may easily be verified that: (a) when n is equal to unity, the equations for stresses and displacements in the top layer are identical to the corresponding ones for the stresses and displacements in the base supporting the top layer.

(b) When r is equal to zero, (S_{rr}) is identical to $(S_{\theta\theta})$.

(c) When n is equal to unity:

$$N = -16(1 - \mu)^2 e^{2mt}$$

$$J_o(0) = 1$$

$$\lim_{x \to 0} \frac{J_1(x)}{x} = \frac{1}{2}$$

$$\int_0^\infty e^{-ax}\, J_1(bx)\, dx = \frac{1}{b} \left[1 - \frac{a}{\sqrt{a^2 + b^2}} \right]$$

$$\int_0^\infty x e^{-ax} J_1(bx)\, dx = \frac{a}{(a^2 + b^2)^{3/2}}$$

$$\int_0^\infty \frac{J_o(ax)}{x}\, dx = \frac{1}{a} \qquad (10)$$

The maximum deflection on the surface is:

$$W(n = 1,\ r = 0,\ z = -t.) = \frac{2(1 - \mu^2)\ at\ p}{E} \int_0^\infty \frac{J_1(mat)}{m}\, dm$$
$$= \frac{2(1 - \mu^2)\ at\ p}{E}$$

Along the axis of symmetry (the vertical axis), the stress components are:

$$S_{zz}(n = 1,\ r = 0,\ z = 0.) = atp \int_0^\infty (1 - mt)e^{-mt} J_1(atm)\, dm$$
$$= p \left[1 - \frac{1}{(a^2 + 1)^{3/2}} \right]$$

$$Srr(n = 1, r = 0, z = 0) = \frac{atp}{2} \quad (mt - 1 \quad - 2\mu)e^{-mt} J_1(atp) dm$$

$$= -\frac{p}{2} \left[1 + 2\mu - \frac{2(1 + \mu)}{(a^2 + 1)^{\frac{1}{2}}} + \frac{1}{(a^2 + 1)^{3/2}} \right]$$

These results check with Boussinesq's solution (28). Also see Timoshenko (23).

Numerical Approximations

<u>Stresses and displacement at points along the axis of symmetry</u>

In general, the functions used in the solution of this problem can be expressed in the form

$$\int_0^\infty F(z,\mu,n,m,t) J_0(mr) J_1(atm)\, dm$$

or

$$\int_0^\infty F(z,\mu,n,m,t) J_1(mr) J_1(atm)\, dm$$

Due to the limitation of the integration formulas and tables available, it is very difficult to evaluate these expressions. However, by making $r = 0$, that is to say, if the solution is limited to evaluating the stress and displacement along the axis only, the terms $J_0(mr)$ and $J_1(mr)$ become constant and can be taken out of the integrand. Now the problem is reduced to the evaluation of the definite integral:

$$\int_0^\infty F(z,\mu,n,m,t) J_1(atm)\, dm.$$

Using the theory of similitude, if z is expressed in terms of t, the integral can be further reduced into

$$F(z,w,n,m)J_1(am)dm.$$

In all cases the function F is still too complicated to find using any available integration formula. A semi-numerical method is employed to get the numerical value of such integrals. This is done in the following steps:

First, an asymptotic expression of the function F is obtained by dividing the term of highest order in the numerator by the term of highest order in the denominator.

Second, some numerical values of the function F are computed for selected values of m.

Third, the numerical values of the asymptotic expression for the same values of m are computed.

Fourth, the difference between the numerical values of the function F and the asymptotic expression are computed for various values of m. Let them equal $Ae^{-am} + Be^{-bm} + Ce^{-cm}$........, where a, b, c, ... are all of higher order than the asymptotic expression. The values of A, B, C... are found from the simultaneous equations.

Fifth, the approximate expression F_1 which consists of the asymptotic expression of the function F and the correcting terms $Ae^{-am} + Be^{-bm}$.. etc., will match the original function at the selected points and at infinity. This approximate expression F_1 is used in the integrand instead of the function F and the approximate numerical value of the integral can be obtained.

This method is illustrated by the following examples.

$$(S_{rr})_1\ (z = r = 0) = ap \int_0^\infty \frac{J_1(ma)}{N} \Big\{ \Big[(3 - 4\mu + n)(4\mu n - 2 - 2\mu)m + 8n(1-\mu)(1 - 2\mu^2 - \mu n) \Big] e^{m} + 2(1-n) \Big[(2\mu n + 3 - 4\mu^2 - \mu)m - 4\mu n(1-\mu) \Big] e^{-m} \Big\} dm = ap \int_0^\infty F(m)\ J_1(ma)dm.$$

Due to the fact that:

$$\lim_{\substack{m \to 0 \\ n \to 0}} F(m)\ J_1(ma) \neq \lim_{\substack{n \to 0 \\ m \to 0}} F(m) J_1(ma)$$

Care must be taken so that the approximate function will represent the function F(m) very closely at points near to the origin especially when n is very small. For example: When $n = 1/800$ and $\mu = 0.2$, the asymptotic expression of the function F is $(2.391m - 0.00333)e^{-m}$. The approximate function F_1 is to match the original function at points $m = 0$, $m = \frac{1}{4}$, and $m = 1$ and F_1 is to vanish at the same value of m as where F vanishes.

The numerical value of F:

$F(0) = 360.3$

$F(\frac{1}{4}) = 44$

$F(1) = 3.62$

$F(m_o) = 0$, $m_o = 0.0233$.

To match these points, we find:

$$F_1(m) = (2.391m - 0.00333)e^{-m} - 469e^{-64m} + 108.6e^{-3.68m}$$

Use the integration formulas (10), the approximate value of

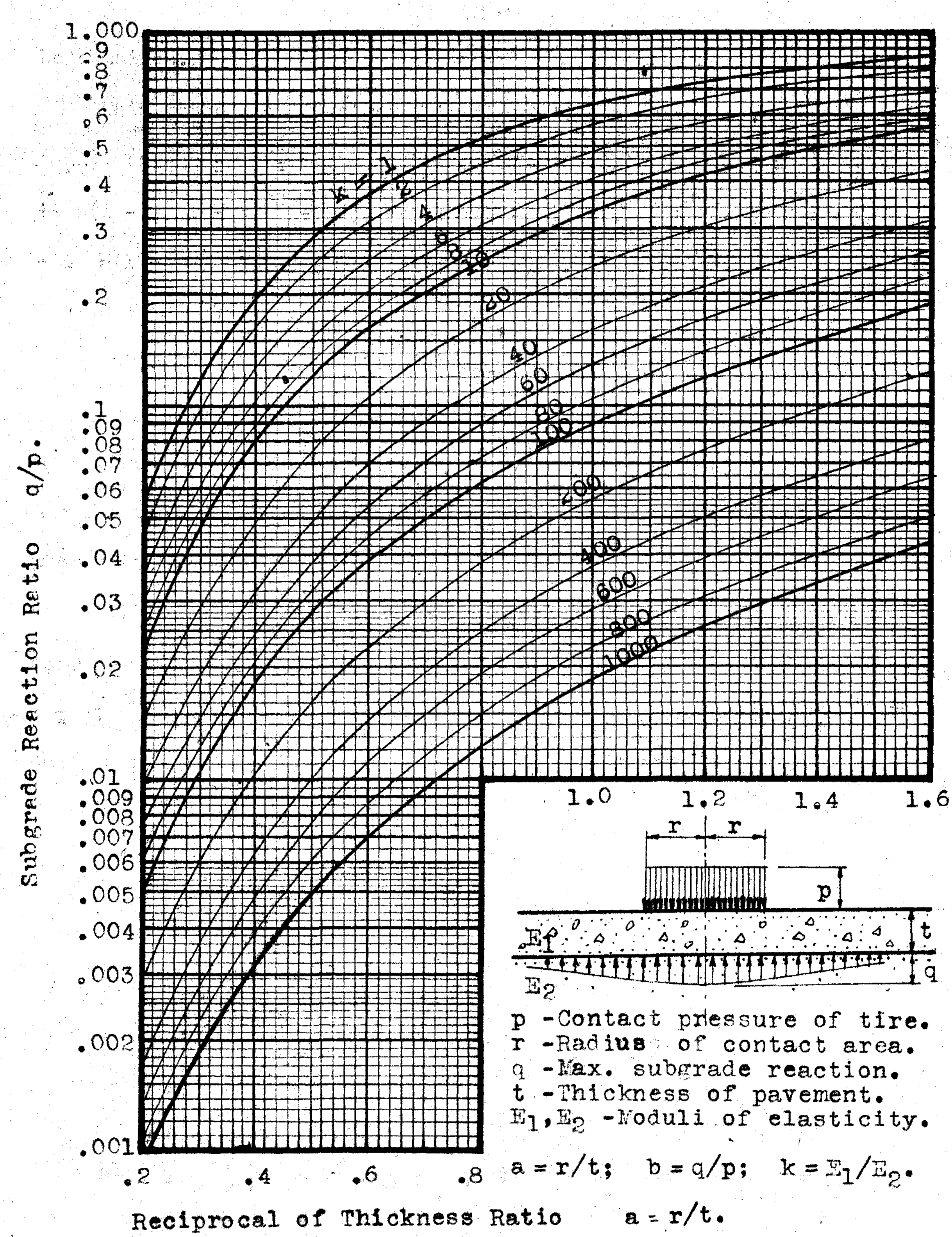


Fig.3a Maximum Subgrade Reaction for Various Load and Pavement Conditions.

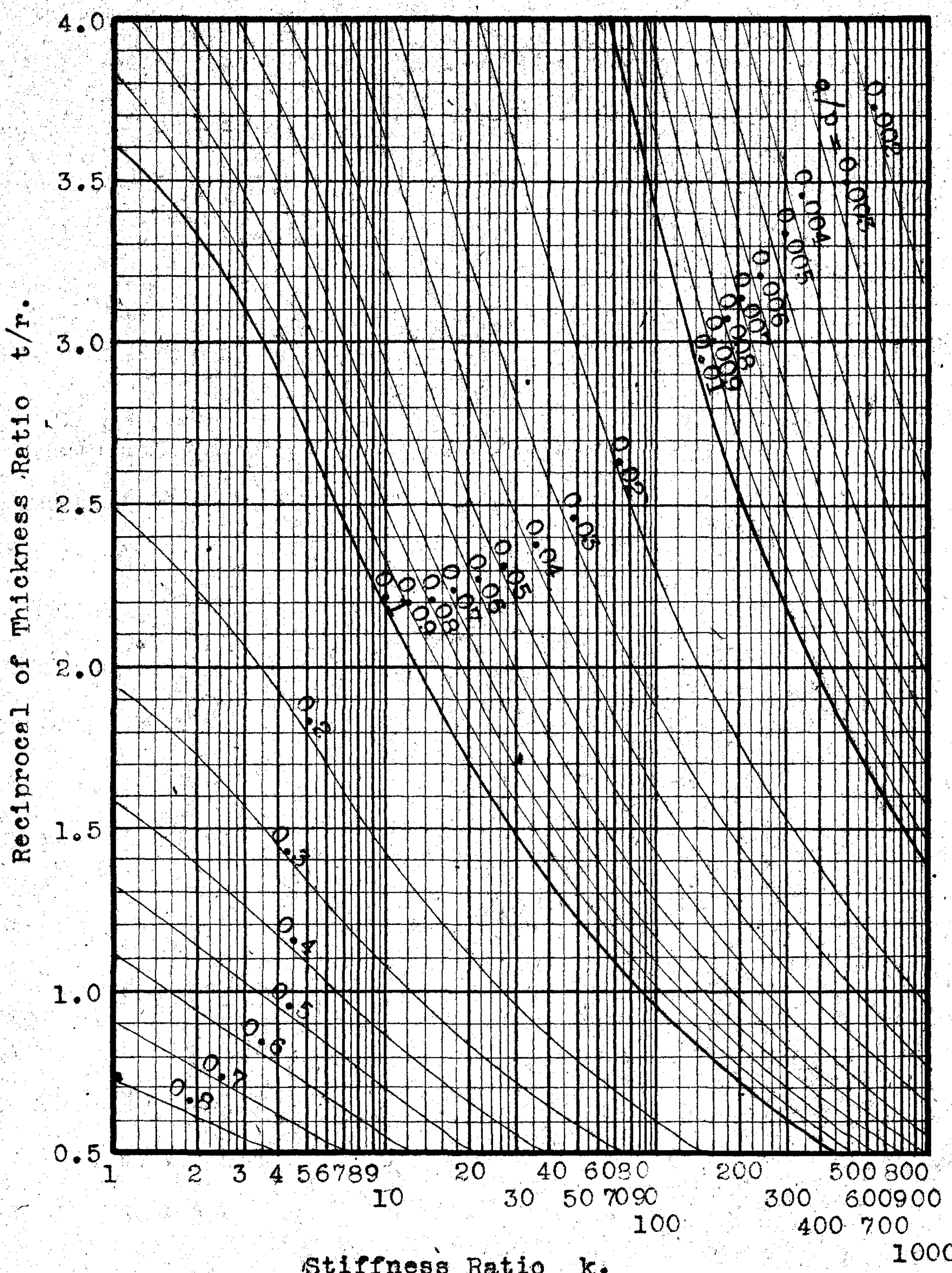


Fig.3b Maximum Subgrade Reaction for Various Load and Pavement Conditions.

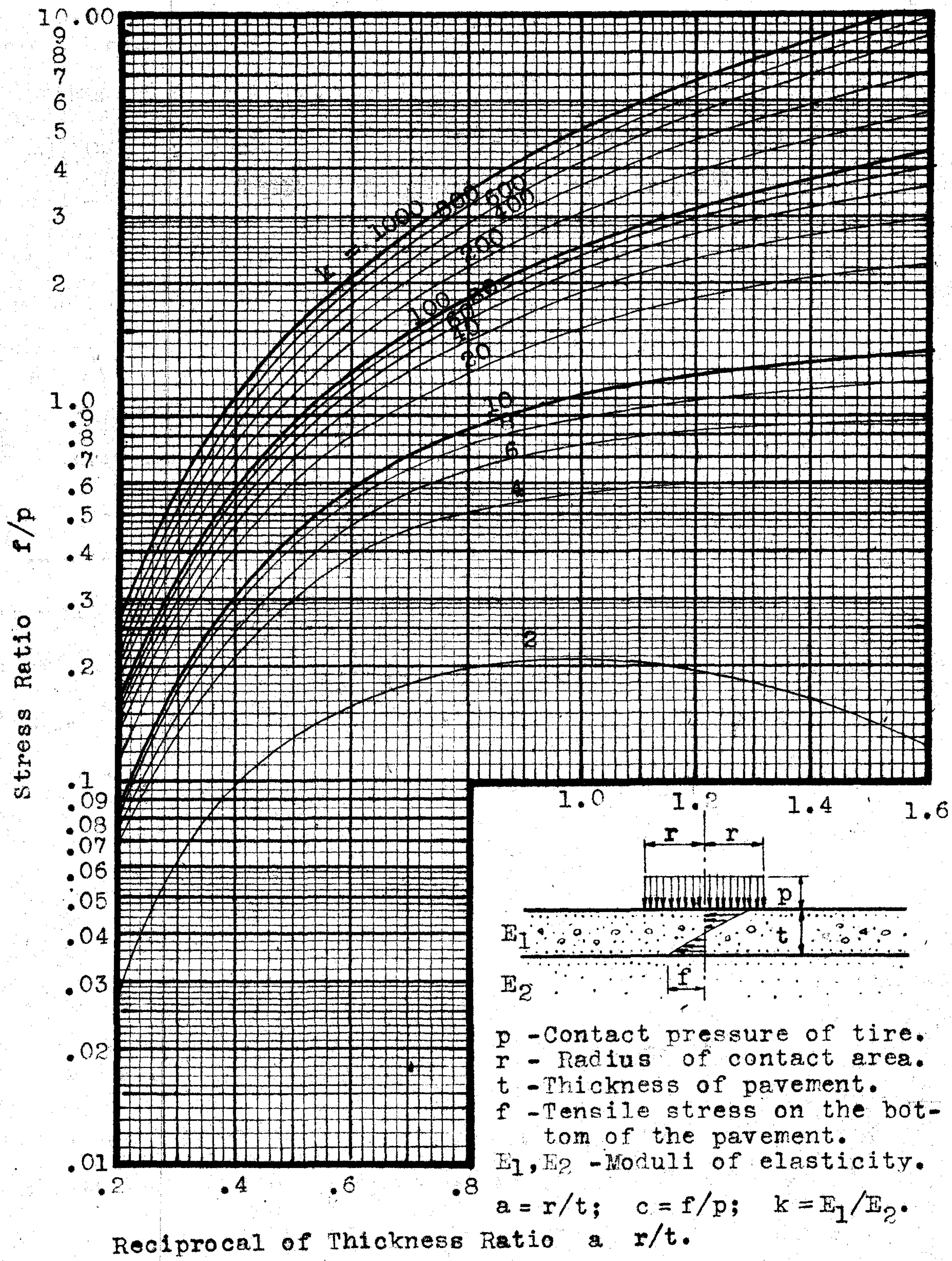


Fig.4a Flexural Stresses in Pavements for Various Load and Pavement Conditions

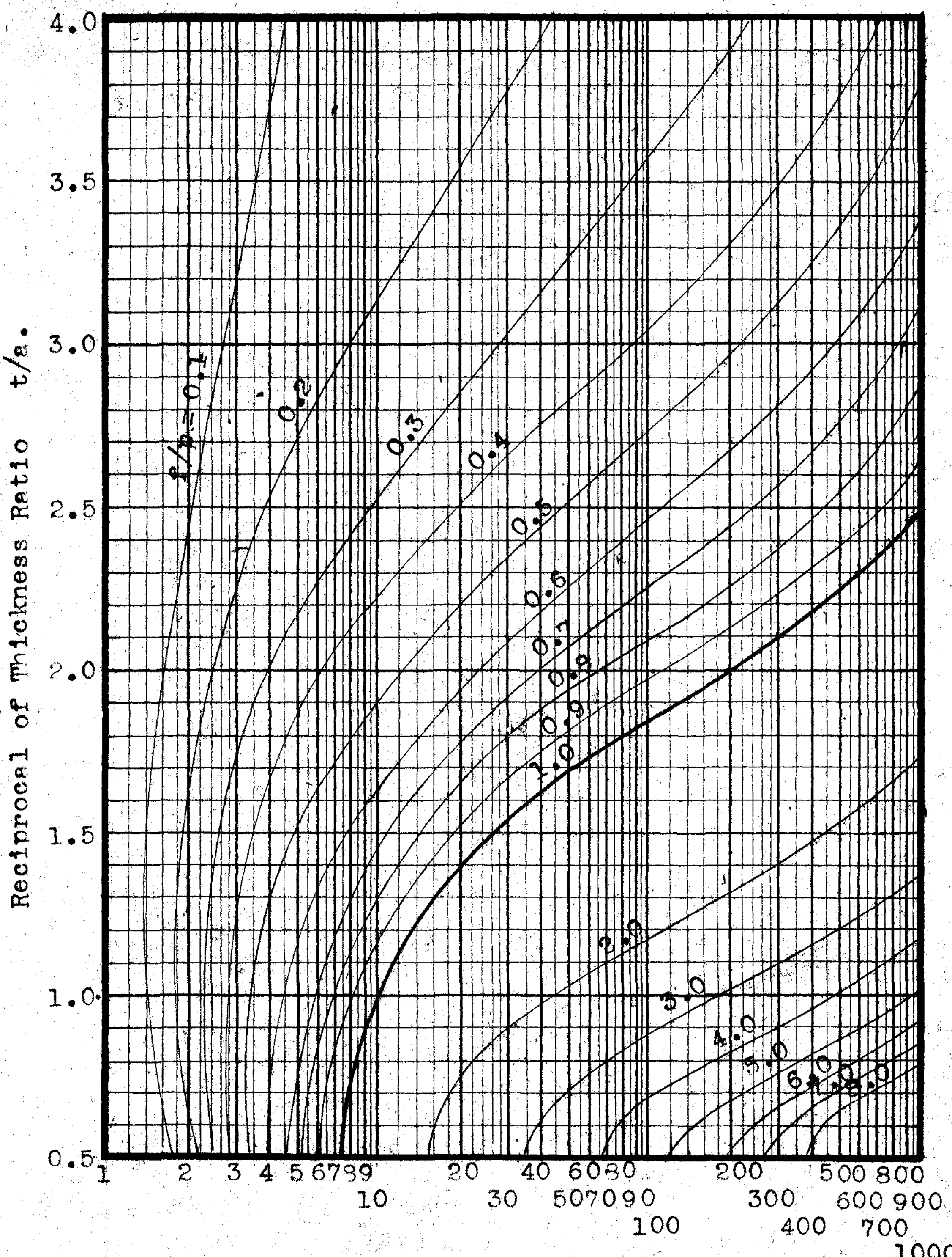


Fig.4_b Flexural Stresses in Pavements for Various Load and Pavement Conditions

$$\frac{(S_{rr})_1}{p}\ (r = z = 0) = 2.391\ \text{vers}\phi - 0.00333\cos\phi\sin^2\phi - 469\ \text{vers}\phi_1$$
$$+\ 108.6\ \text{vers}\phi_2$$

in which: $\phi = \tan^{-1} a$; $\phi_1 = \tan^{-1}\frac{a}{64.0}$; $\phi_2 = \tan^{-1}\frac{a}{3.68}$

These expressions can all be evaluated quite readily numerically.

The numerical value of the other stresses along the axis can be computed in the same way. For the convenience of the highway design engineer, the numerical values of $(S_{zz})_{(r = z = 0)}$ and $(S_{rr})_{1(r=z=0)}$ are computed and plotted as Charts 3a, 3b, 4a, and 4b, which give the maximum subgrade reaction and the maximum flexural stresses in the pavement respectively for various load and pavement conditions.

<u>Stresses and displacements at points not on the axis of symmetry</u>

Consider the Fourier-Bessel integral:

$$\int_0^\infty dm \int_a^b mx\ F(x)\ J_n(mx)\ J_n(mr)\ dx = \begin{cases} F(r), & \text{if } a < r < b; \\ 0, & \text{if } r > b \text{ or} < a. \end{cases}$$

This function has discontinuities at $r = a$, and at $r = b$. Although the integral exists for all values of r, the convergence of the infinite integral is very poor when r gets close to either a or b. To overcome this handicap and to make numerical integration possible, the infinite integrals involved in this study are reduced into proper definite integrals.

Since:

$$J_1(ma) = \frac{1}{\pi}\int_0^{\pi} \sin\ (ma\sin\theta)\sin\theta\ d\theta;$$

$$J_o(mr) = \frac{1}{\pi}\int_0^{\pi} \cos\ (mr\sin\phi)\ d\phi.$$

By changing the order of integration, we get:

$$F_1(a,r) = \int_0^{\infty} e^{-m}J_1(am)J_o(rm)dm = \frac{1}{2\pi^2}\int_0^{\pi} d\theta \int_0^{2\pi} d\phi\ \frac{\sin\theta(a\sin\theta + r\sin\phi)}{1 + (a\sin\theta + r\sin\phi)^2};$$

$$F_2(a,r) = \int_0^{\infty} me^{-m}J_1(am)J_o(rm)dm = \frac{1}{\pi^2}\int_0^{\pi} d\theta \int_0^{2\pi} d\phi\ \frac{\sin\theta(a\sin\theta + r\sin\phi)}{\left[1 + (a\sin\theta + r\sin o)^2\right]^2};$$

$$F_3(a,r) = \int_0^{\infty} me^{-m}J_1(am)J_1(rm)dm = \frac{1}{2\pi^2}\int_0^{\pi} d\theta \int_0^{2\pi} d\phi\ \frac{\sin\theta\sin\phi\left[1 - (a\sin\theta + r\sin\phi)^2\right]}{\left[1 + (a\sin\theta + r\sin\phi)^2\right]^2}$$

$$F_4(a,r) = \int_0^{\infty} e^{-m}J_1(am)J_1(rm)dm = \frac{-1}{2\pi^2}\int_0^{\pi} d\theta \int_0^{2\pi} d\phi\ \frac{\sin\theta\sin\phi}{\left[1 + (a\sin\theta + r\sin\phi)^2\right]^2};$$

$$F_5(a,r) = \int_0^{\infty} \frac{e^{-m}}{m}J_1(am)J_1(rm)dm = \frac{1}{4\pi^2}\int_0^{\pi} d\theta \int_0^{2\pi} d\phi\ \sin\theta\sin\phi\ \log_e \left[1 + (a\sin\theta + r\sin\phi)^2\right] \quad (11)$$

For functions involving higher exponential powers:

$$\int_0^{\infty} e^{-am}J_1(bm)J_o(cm)dm = \frac{1}{a}F_1\left(\frac{b}{a},\frac{c}{a}\right);$$

$$\int_0^{\infty} me^{-am}J_1(bm)J_o(cm)dm = \frac{1}{a^2}F_2\left(\frac{b}{a}\ \frac{c}{a}\right);$$

$$\int_0^{\infty} me^{-am}J_1(bm)J_1(cm)dm = \frac{1}{a^2}F_3\left(\frac{b}{a}\ \frac{c}{a}\right);$$

$$\int_0^{\infty} me^{-am}J_1(bm)J_1(cm)dm = \frac{1}{a}F_4\left(\frac{b}{a}\ \frac{c}{a}\right);$$

$$\int_0^{\infty} \frac{e^{-am}}{m}J_1(bm)J_1(cm)dm = F_5\left(\frac{b}{a}\ \frac{c}{a}\right). \quad (11a)$$

The stress at any point can be computed in general by the following steps:

$$S = ap\int_0^\infty F(m, z, k, \mu)J_1(am)J_n(rm)dm.$$

First step: For any given set of values of z, k and v, compute the function $F(m,z,k,\mu)$ numerically at some selected points for different values of m.

Second step: Find an approximate function of F in terms of exponential forms.

Third step: Integrate the approximate function numerically by the formulas (11) and (11a).

For example, to find the shearing stress under a pavement of stiffness ratio equal to 100, Poisson's ratio equal to 0.2 with the radius of the loaded area equal to at, the stress function is given by equation (9a).

For these given values of z (equal to zero), k, and μ, the function F is computed:

at $m = 0$, $F = 0$;

at $m = 0.5$, $F = 0.196$

at $m = 1.0$, $F = 0.0585$

at $m = 2.0$, $F = 0.0143$

at $m = 3.0$, $F = 0.00572$

The approximate function for F(m) is

$F(m) = 0.0313me^{-m} + 0.1422me^{-2m} + 3.715me^{-4.90m}$ approximately.

The first term is the asymptotic expression of the function; the second and third terms are to correct the numerical value of the asymptotic expression of the function; the second and third terms are to correct the numerical value of the asymptotic expression for small values of m. The corresponding values of the approximate function for the points m equal to 0, 0.5, 1.0, 2.0 and 3.0 are .196, .0585, .0144 and .00572, respectively. They are very close to the values of the exact function.

The shearing stress under the pavement is

$$S_{rz}\ (z = 0,\ k = 100,\ \mu = 0.2) = ap\left[0.0313F_3(a,r) + 0.0374F_3(0.5a,\ 0.5r) + 0.1547F_3(0.204a, 0.204r)\right].$$

Where r is expressed in terms of the thickness of the pavement.

To illustrate the effect of the stiffness ratio of the pavement on the distribution of the stress, an example in which $a = 1$, $\mu = 0.2$, $z = 0$, $k = 1$, 10, 100 and 1,000, respectively, is actually computed and plotted as shown in Figure 5. It should be observed that: (1) the maximum subgrade reaction decreases when the stiffness ratio increases; (2) the spreading of the vertical pressure increases with the stiffness ratio; (3) the shearing stress under the pavement decreases when the stiffness ratio increases.

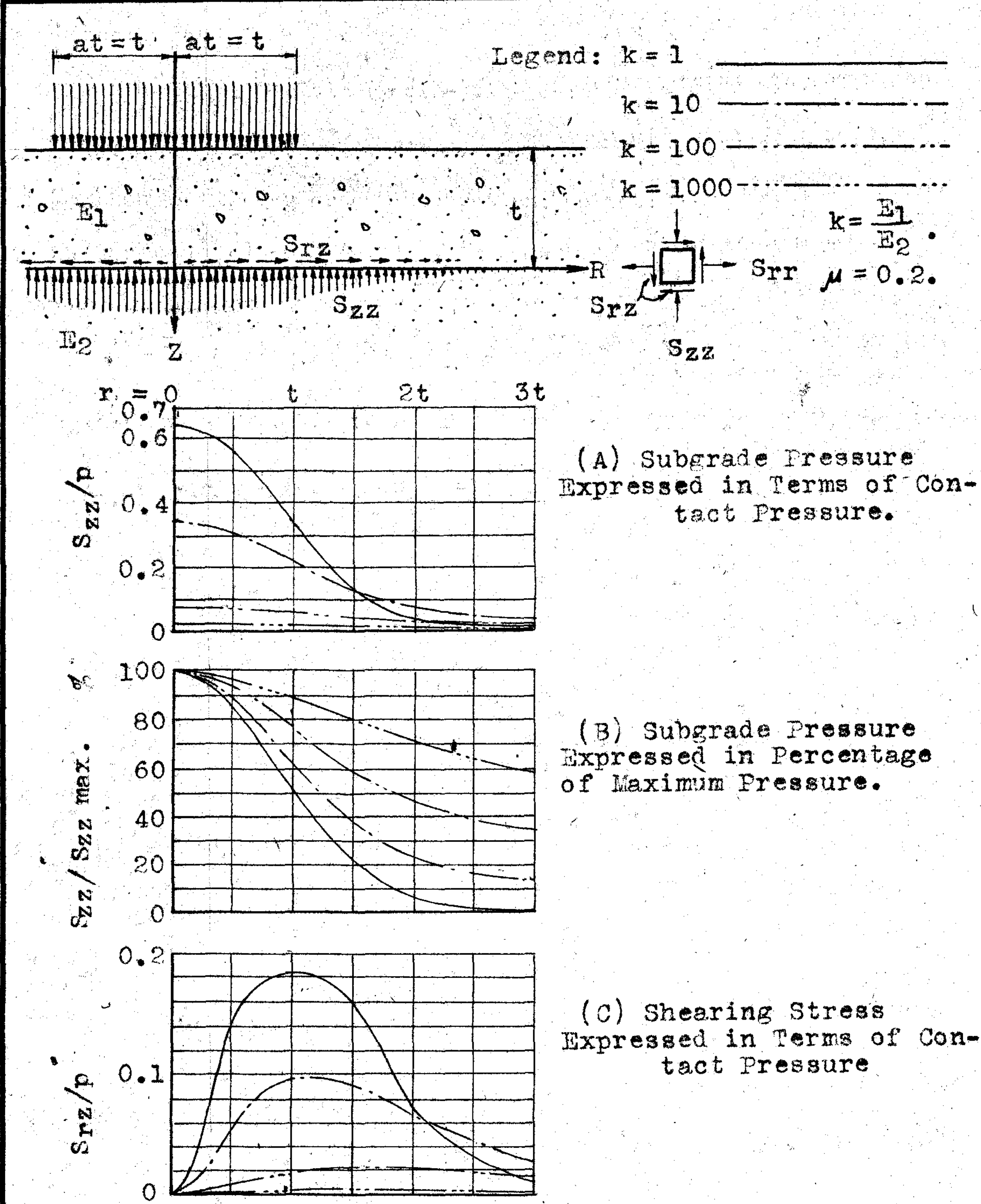


Fig.5 Effect of Stiffness Ratio of the Pavement on the Stresses at the Interface.

APPLICATIONS TO HIGHWAY PAVEMENT DESIGN

General Discussion

Assumptions made in this part of the investigation

In making practical applications of the results of the analysis, one must fully understand the limitations of the ideal conditions or concepts which form the basis of the assumptions.

1. Assumptions regarding contact area of tires. It is known that the distribution of load under a pneumatic tire is not uniform and the shape of load of the tire imprint tends to be elliptical rather than circular and for certain loads it may even be nearly rectangular. Nevertheless, by Saint Venant's principle, (See Timoshenko (23), p. 31.) the assumption of a circular uniform loading equivalent to the measured wheel load will lead to no serious error in computing the stresses at points such as at the interface which is an appreciable distance away from the loaded region. Greater care must be taken when using these formulas to determine the stresses at points very close to the contact area.

2. Assumption regarding the elasticity of the materials. Objections have been directed to the treating of soil and bituminous mixtures as elastic materials. However, in flexible pavement design it is preferred to design the pavement to develop only small amount of deformation under the wheel load; in that case, all the materials involved may be treated

as elastic. At the same time, the results will not be representative when the pavement is failing because plastic flow already has been taking place.

3. Assumption of the infinite layer. With the limitation of this assumption, the results can be applied when the wheel load is applied only at the interior points of the pavement. As shown by the numerical example (Figure 5), for flexible pavements the stresses at points at the interface three times the radius of the contact area away from the axis of the load will be negligible. For rigid pavement, the infinite layer assumption will result in higher values of flexural stress and lower values of subgrade reaction than the corresponding values in an actual pavement. The present trend in highway and airport pavement design is to use wider pavements of uniform thickness. The occurrence of a heavy wheel load close to the edge of the wider pavements will be infrequent. By adopting suitable factors of safety for the allowable stresses and by making some adjustment of the edge thickness, a satisfactory design of the pavement can be obtained by the method developed in this study.

4. Assumption of perfect continuity at the interface. In criticising Burmister's analysis (2), Professor Casagrande stated:

> The assumption of perfect continuity at the interface may be such on the unsafe side. The greater the difference in resistance to deformation between the two layers, the greater are the shearing stresses theoretically transmitted at the interface (e.g. concrete pavement directly on clay). But if these computed stresses approach the shearing strength of the softer material, then the maximum normal stress on

> the soft material will become much greater than the computed value....

Actually the computed results of the shearing stresses at the interface have not been published heretofore. The results of this study show (Figure 5) on the contrary, that the shearing stresses at the interface decrease as the relative stiffness increases. There is no reason why the subgrade should not have enough strength to take care of the maximum shearing stresses under a rigid pavement, because these stresses are only a fraction of similar stresses under a flexible pavement.

Criteria of design

In the author's opinion, the best criteria of pavement design are: for flexible pavements, the bearing stress or load transmitted by the paving material and the allowable subgrade support; for rigid pavements, the flexural strength of the slab. The author recognizes the desirability of using the settlement of the pavement as the governing factor in the design; however, he cannot quite agree with Burmister's method. The theoretical value of the settlement of the surface of a pavement is an infinite integral of the vertical strain. By Saint Venant's principle, the stress at a point will be only slightly affected by replacing a force system at region sufficiently away from the said point by another equivalent force system. The differences between the ideal conditions in the assumptions and the actual conditions such as for the dimensions and intensity of the load, and the elastic property of the materials will have only a slight effect on the results of the stresses analysis, but may result in large errors in the computed deflections.

There is little controversy concerning the method of measuring the flexural strength of concrete. However, the determination of the strength of bituminous mixtures, stabilized soil mixtures and soil presents an entirely different picture. Many testing methods to determine the "quality" of a bituminous or a stabilized soil mixture or a subgrade soil have been developed. But if one is interested in making a structural analysis of flexible pavement in which the bearing strength of the pavement and the subgrade capacity of the soil are important factors in solving the problem, there is no agreement in regard to the testing method and the value of bearing strength to use.

In recent years, it has been common practice to express the strength properties of bituminous or soil mixtures in terms of their cohesive strength and the angle of internal friction. However, in the design of flexible pavements this method of analysis is not satisfactory. Usually the flexural stresses in a flexible pavement may be assumed to be small and can be neglected, then the stress distribution near the top surface of the pavement will be approximately the same as that in a homogeneous semi-infinite body under the same load. The critical points where the failure in bearing stress will start are along the edge of the loaded area. According to Love (28), the limits of the stresses at these points depend upon the angle of approaching:

$$S_{rr} = \frac{p}{2\pi}\left[(1 - 2\mu)\pi - 2i - \sin 2i\right]$$

$$S_{zz} = \frac{p}{2\pi}\ (-2i + \sin 2i)$$

$$S_{rz} = \frac{-1}{\pi}\sin^2 i$$

in which the stresses considered are for points on the surface along the circumference of the circular loaded area and i is the angle of approaching. Since the vertical section directly under the circumference is the critical section, let i be equal to $\frac{1}{2}\pi$. Then,

$$S_{rr} = -\mu p$$

$$S_{zz} = -\frac{1}{2}p$$

$$S_{rz} = -p/\pi$$

Using Mohr's theory of failures (Figure 6) cohesion in terms of the intensity of the load and the angle of internal friction of the material may be computed according to the formula:

$$\frac{C}{p} = \frac{\cos\phi\left[\cos\phi\sqrt{(1-2\mu)^2\pi^2 + 16} - (1+2\mu)\pi\right]}{4\pi(\csc\phi - \sin\phi)} \qquad (12)$$

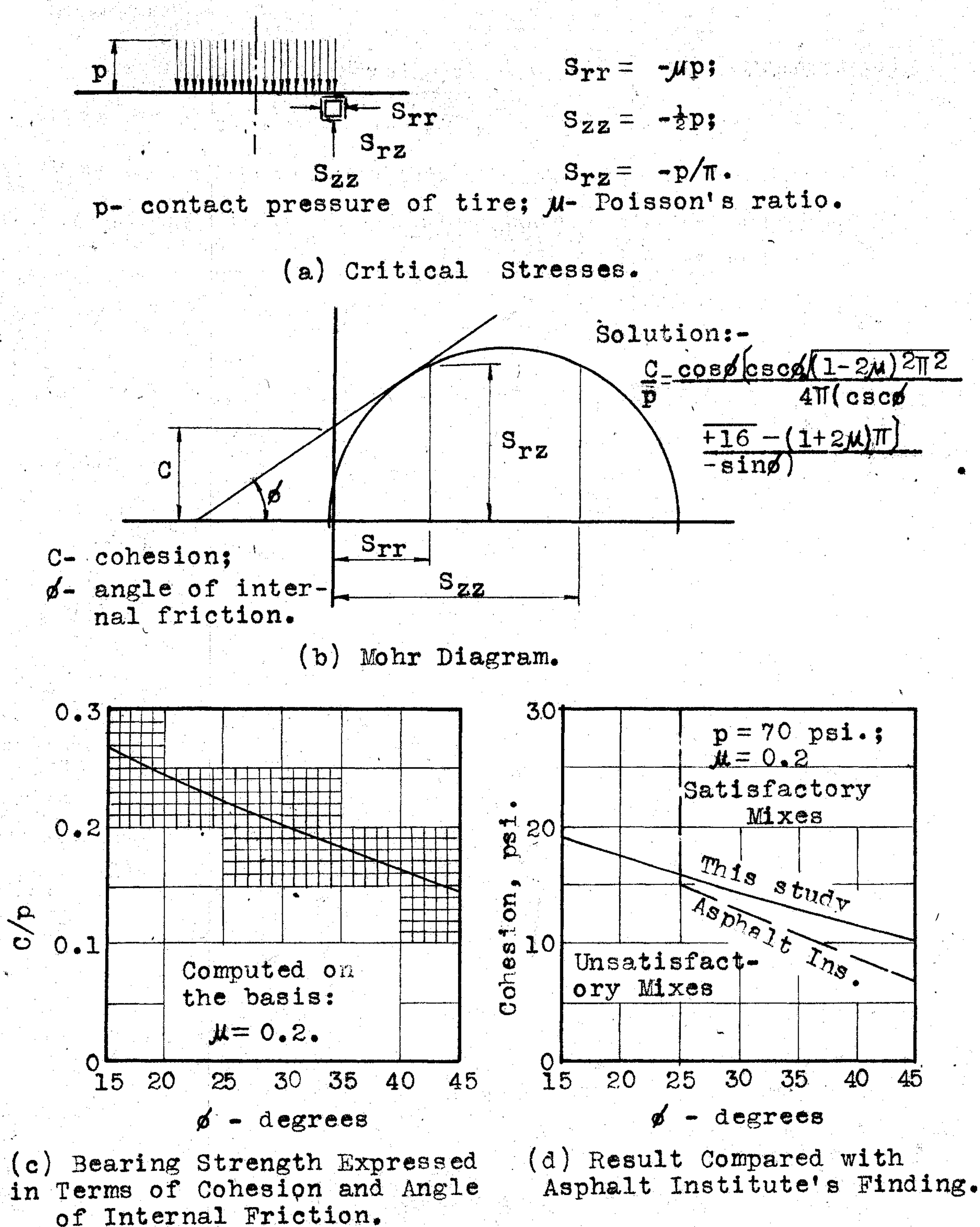
in which C is the required cohesive strength of the material in psi and ϕ is the angle of internal friction.

If μ is equal to 0.2, equation 12 may be reduced to: (12a)

$$\frac{C}{p} = \frac{0.35\cos\phi(\csc\phi - 1)}{\csc\phi - \sin\phi}$$

By substituting appropriate values of C, p, and ϕ in the graphical solution in this formula, Figure 6e is obtained.

If the contact pressure for heavy trucks or airplane wheel loads



$$\frac{C}{p} = \frac{\cos\phi\left[\csc\phi\sqrt{(1-2\mu)^2\pi^2+16}-(1+2\mu)\pi\right]}{4\pi(\csc\phi-\sin\phi)}.$$

Fig.6 Bearing or Load Supporting Strength of Flexible Paving Mixtures.

of 70 pounds per square inch is used, the required cohesive strength of the paving mixture may be computed and plotted as in Figure 6a. The results obtained from tests by the Asphalt Institute (36) are plotted in Figure 6d and compare favorably with the values obtained using formula 12a. It should be noted that due to the cutting edge effect involved in the ideal condition of the circular area loading assumption, the result obtained is on the safe side.

Since it is impossible to obtain a set of simplified formulas to express the stresses at the interface of the pavement and the subgrade, the application of Mohr's theory to evaluate the bearing capacity of the subgrade soil will be very complicated and of no practical value. Therefore the designer must still rely upon the past experience and judgment to select the proper bearing stress value. The subgrade bearing capacity is computed on the basis of the permissible deflection as recommended by Palmer and Barber (8), the following formula may be used:

$$q = \frac{E_2 d}{1.5\, r}$$

in which d is the permissible deflection of the subgrade commonly taken as 0.1 inch and r is the radius of the contact area of the tire. Since the distribution of the wheel load on the subgrade is much wider than it is on the surface of the pavement, the actual values of d for given values of q should be higher than the values computed using this formula.

The results of this analysis show that tensile stresses may occur in the flexible type of pavements. In the discussion of Burmister's paper (2) Casagrande states: ".... the materials (for most flexible pavement) for practical purposes cannot take tension." However, a bituminous mixture generally retains a portion of the lateral thrust caused by heavy vertical load even after the load is removed. It is reasonable to expect that the small amount of tensile stress in a flexible pavement caused by the wheel load will only counteract the lateral precompression in the pavement caused by the heavy rollers during construction. This may explain why a bituminous pavement carrying very light traffic is more likely to break down under a heavy wheel load than a similar pavement carrying heavy traffic. The development of tensile stresses in flexible pavements by precompression methods of construction is an interesting design feature worthy of further study.

Correlation of the modulus of elasticity of the subgrade, the modulus of subgrade reaction, the California bearing ratio and the subgrade bearing capacity

As discussed in previous paragraphs, the determination of acceptable values of modulus of subgrade reaction and subgrade bearing capacity is still debatable. It is impossible to establish a definite relationship between these two terms and the term modulus of elasticity as used in this study. However, using the methods by which these two values are commonly determined by laboratory tests, approximate relationships can be derived. In common practice, a thirty inch diameter rigid plate is used to test the bearing value of soils. The deflection of the surface of an elastic infinite layer under a rigid plate is given by the formula:

$$d = \frac{(1 - \mu^2)rq}{2E_2}$$

in which d is the maximum deflection of the surface in inches;

r is the radius of the rigid plate in inches;

q is the average bearing pressure in p.s.c.;

is the Poisson's ratio of the soil;

E_2 is the modulus of elasticity of the soil in p.s.c.

Using a value of r equal to 15 inches and μ equal to 0.2, the formula may be reduced to:

$$q = 0.0442dE_2$$

By the definition of modulus of subgrade reaction for these conditions:

$$K = \frac{q}{d} = 0.0442E_2 \qquad (13)$$

in which K is the modulus of subgrade reaction in pounds per square inch per inch and E_2 is in pounds per square inch.

If the allowable deflection is 0.1 inch, the allowable bearing capacity of the soil will be:

$$q = 0.00442E_2 \qquad (14)$$

The relationship between the modulus of elasticity of subgrade soil and the California Bearing Ratio can be indirectly established by the formula (12) and the approximate relationship between the California bearing ratio and the K value may then be obtained by the plate bearing method (37) (See Figure 7).

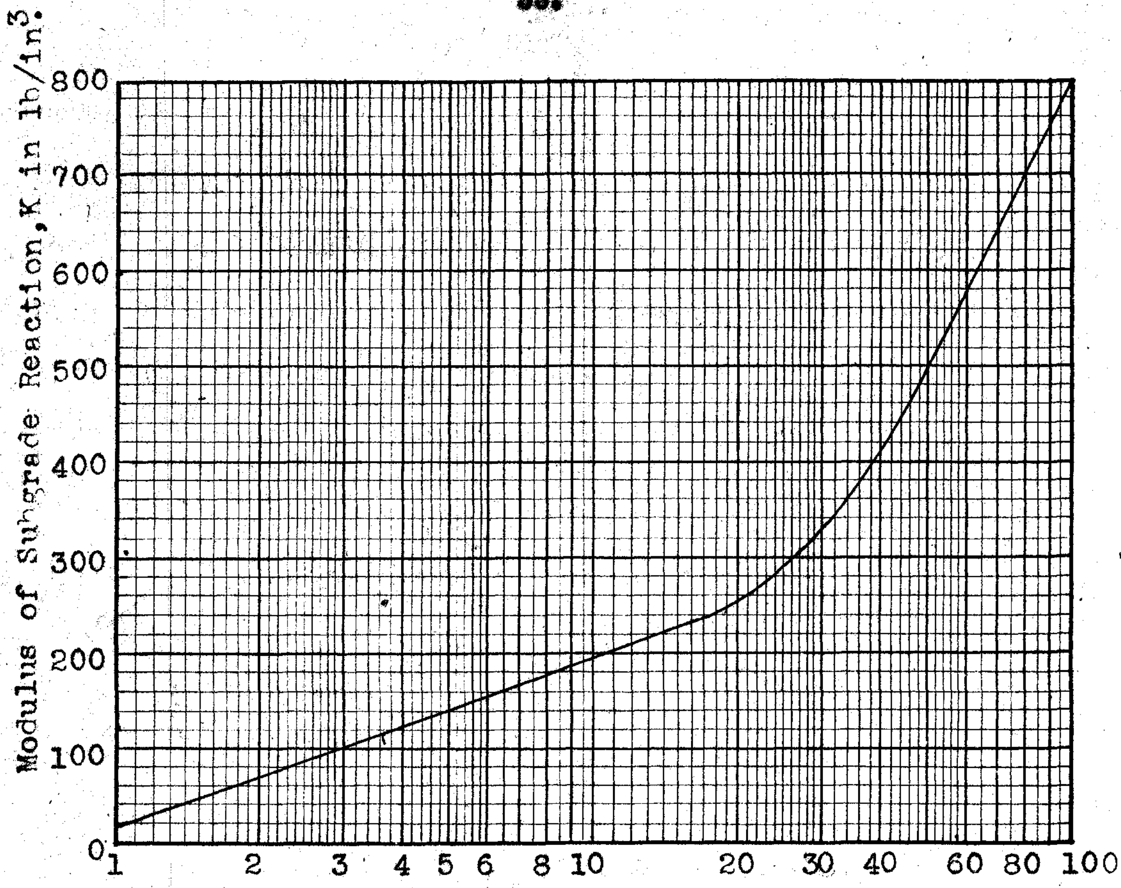


Fig.7* Correlation of the Modulus of Subgrade Reaction and the California Bearing Ratio

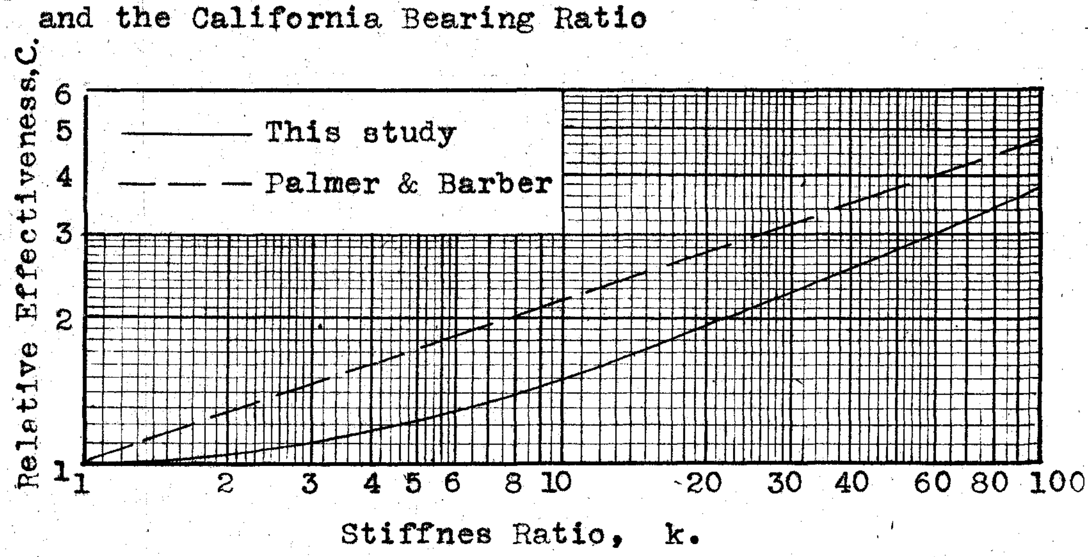


Fig.8 Relative Effectiveness of Flexible Paving Mixtures

* According to Middlebrooks and Bertram (37).

Practical Applications

Approximate design formulas

Approximate formulas to give reasonably accurate results within the practical ranges of materials likely to be used, derived for design purpose in this study are as follows.

For flexible pavements:

$$q = (1 - \cos^3\theta')p$$

$$\theta' = \tan^{-1} r/t'$$

$$t' = \frac{0.7^k + k^{0.4}}{1.7} \quad t = C\,t \qquad (15)$$

in which: q is the maximum subgrade reaction in pounds per square inch; p is the intensity of contact pressure in pounds per square inch; t is the thickness of the pavement in inches, t' is the equivalent thickness in inches; r is the radius of contact area in inches; C is the relative effectiveness of the paving material; and k is the stiffness ratio. The usable range of formula (15) is for values of k equal to unity up to about 200. Based on their experiments on the deflection of flexible pavements, Palmer and Barber recommend that the required thickness of a pavement should be inversely proportional to the cube root of the modulus of the pavement. In other words they probably have in mind that the relative effectiveness of flexible pavements should be equal to the cube roots of their stiffness ratio although Palmer and Barber did not use these terms. To illustrate this point, the values of C for different values of k according to this study and according to Palmer and Barber's idea are plotted in Figure 8. It should be observed that if both curves are started at $k = 3$ which is practically the lowest value of k for a flexible pavement, these two curves agree with one another fairly closely.

For rigid pavements:

$$f = 0.625pa^2 \left[k^{0.301} - 1.3(a^2 - a) \right] ; \qquad (16)$$

$$a = \frac{r}{t} .$$

in which f is the flexural stress in the pavement; k, p, r, and t have the same meaning as in the flexible pavement formula (15). The usable range of this approximate formula is for values of stiffness ratio ranging from 200 to 5,000.

Comparison of the results with other formulas

Since this analysis is based upon factors somewhat different than those used by other investigators, a general comparison of the results of this analysis with those of others cannot be made in all cases. In the illustrative examples, which follow, a comparison of the results obtained in this study with the results obtained by the California Bearing Ratio method and by using Westergaard's formula are tabulated. The flexural stresses due to wheel loads at interior points of a concrete slab obtained from this study are in most cases higher than those computed according to Westergaard's formula (See Table on page 65).

The comparison of the results obtained in this study with the results obtained using Downs formula, the Asphalt Institute formula for flexible pavements, and with Older's formula for the interior thickness for concrete pavements. In making this comparison, these formulas must first be reduced into comparable form:

Downs formula:

$$t = \sqrt{\frac{W}{\pi q}}.$$

$$W = \pi r^2 p.$$

Substitute W into the formula,

$$t = \sqrt{\frac{r^2 p}{q}};$$

$$\frac{r^2}{t^2} = \frac{p}{q}$$

Asphalt Institute formula:

$$t = \sqrt{\frac{W}{q}} - r = \sqrt{\frac{r^2 p}{q}} - r;$$

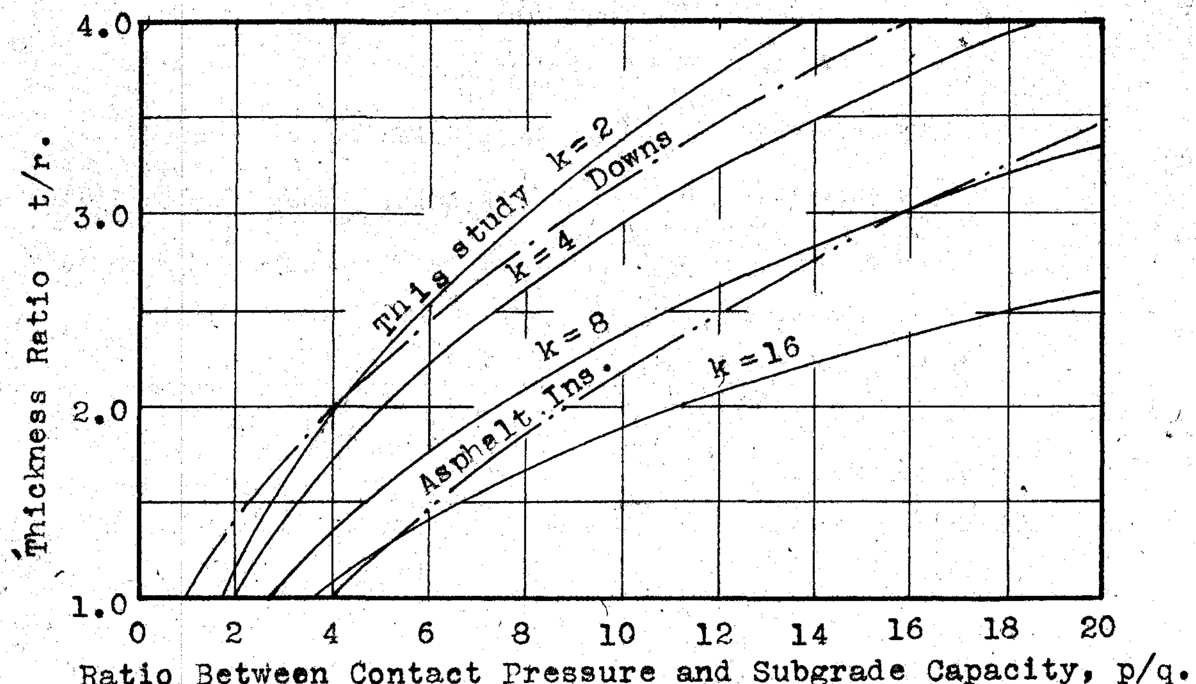
$$\frac{(t + r)^2}{r^2} = (1 + \frac{t}{r})^2 = \frac{p}{q}$$

Older's formula for the interior thickness of concrete pavement assuming the load to be distributed over four corners:

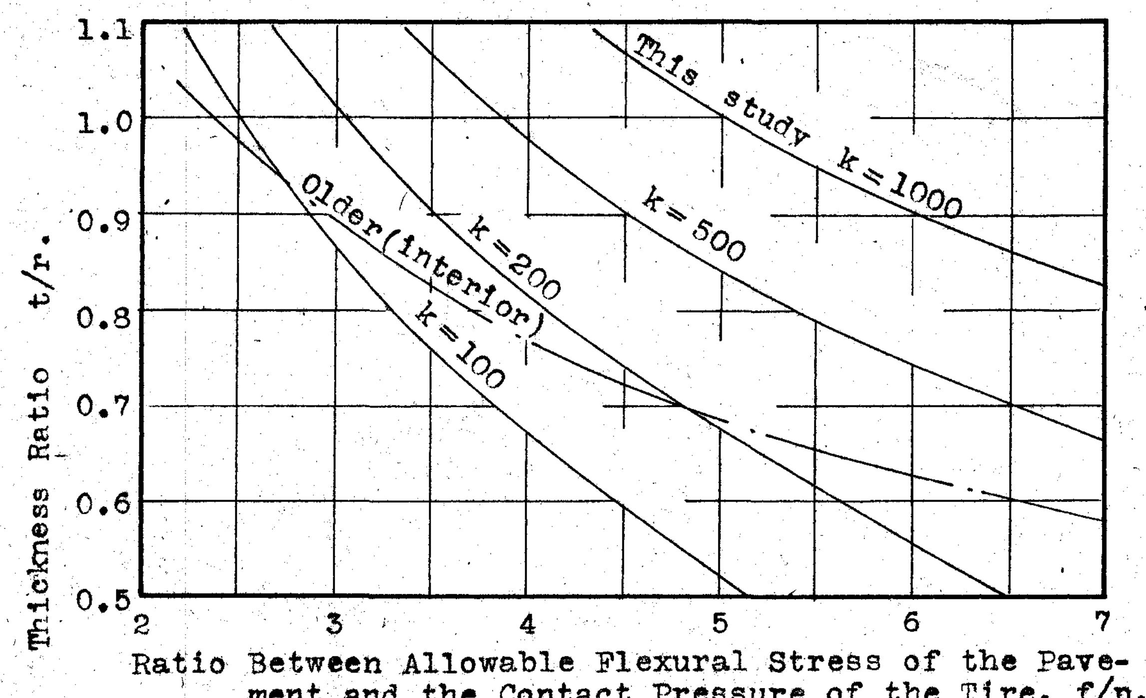
$$t = \sqrt{\frac{0.75W}{f}} = \sqrt{\frac{0.75 \pi r^2 p}{f}}$$

$$\frac{t^2}{r^2} = \frac{0.75 \pi p}{f} = \frac{2.36p}{f}$$

It should be noted that the results of this study agree with the



A. For Flexible Pavements



B. For Rigid Pavements

Fig. 9 Comparison of the Results Obtained in this Analysis with the Results Obtained Using Other Design Formulas.

opinion held by certain designers of pavements that proper concentration factors should be used in the design formulas now widely used for flexible pavements.

<u>Proposed procedure for "structural design" of flexible pavements</u>

The author proposed the following procedure for flexible pavement design which is based upon the principle that the elements of a flexible pavement should be determined entirely on the basis of stress analysis.

Step one, the strength properties of the materials involved in the given design to be determined by laboratory tests are: the moduli of elasticity of the paving mixture and the subgrade soil, the cohesive strength and the angle of internal friction of the paving mixture, and the allowable bearing capacity of the subgrade. The modulus of elasticity can be determined by the unconfined compression test. The cohesive strength and angle of internal friction of a material can be determined by the triaxial compression test or by a suitable shear test. The bearing capacity of the subgrade, for the time being when a more satisfactory method is not available, can be determined by a bearing test using a plate with a diameter not less than thirty inches or by formula (14).

Step two, proper values for the total wheel load and contact pressure of the tire are selected according to the type of traffic and wheel loads expected. The radius of the contact area is calculated by assuming a uniform distribution of contact pressure over a circular area.

Step three, the bearing strength of the paving mixture is checked with formula (12) or Figure 6c.

Step four, the required thickness of the pavement is determined from the subgrade reaction ratio as given in Figure 3_bor 3a, or by a cut-and-try method using formula (15).

Step five, the edge of the pavement should be designed so that, either by widening the pavement or by spreading the footing, the bottom edge of the pavement will be at least three times the radius of the tire contact area from the edge position of the wheel load in normal operation.

Illustrative examples

(1) Determination of maximum subgrade reactions.

Given: Wheel load w = 10,000 lb.

Contact pressure of tire p = 60 psi.

Modulus of elasticity of compacted subgrade, E_2 = 4,000 psi.

Thickness of pavements, t = 12 in.

To find the maximum subgrade reaction under different types of pavements with the following values of moduli of elasticity:

Sand clay mixture, E_1 = 8,000 psi;

Gravel mixture, E_1 = 12,000 psi;

Bituminous mixture A, E_1 = 16,000 psi;

Bituminous mixture B, E_1 = 20,000 psi;

Bituminous mixture C, E_1 = 24,000 psi.

Solution:

The radius of contact area of tire, $r = \sqrt{\frac{W}{\pi p}} = \sqrt{\frac{10,000}{60\pi}} = 7.28$ in.

$r/t = 7.28/12 = 0.607$.

Material of pavement:	Sand-clay	Gravel	Bituminous mixture A	Bituminous mixture B	Bituminous mixture C
$k = \frac{E_1}{E_2}$	2	3	4	5	6
q/p from Fig. 3a	0.32	0.28	0.25	0.23	0.21
Subgrade reaction, q	19	17	15	14	13 psi.

(2) Flexible pavement design.

Given: Wheel load, W = 8,000 lb.

Contact pressure of tire, p = 55 psi.

Modulus of elasticity of an A-7 subgrade soil, $E_2 = 2,000$ psi.

Bearing capacity of the subgrade soil, q = 10 psi.

Modulus of elasticity of gravel mixture, $E_1 = 10,000$ psi.

Modulus of elasticity of asphaltic concrete, $E_1 = 20,000$ psi.

It is proposed to determine the required thickness of these two different types of flexible pavements and to compare the results with those obtained using the California Bearing Ratio, and Barber's, Downs, and the Asphalt Institute formulas.

Solution:

$$r = \sqrt{\frac{W}{\pi q}} = \sqrt{\frac{8,000}{55\pi}} = 6.80 \text{ in.}$$

$$q/p = 10/55 = 0.182.$$

For the gravel mixture k = 10,000/2,000 = 5.

From Figure 3a, r/t = 0.54

$$t = 6.80/0.54 = 12.6 \text{ in.}$$

For the asphaltic concrete, k = 20,000/2,000 = 10

From Figure 3a, r/t = 6.5.

$$t = 6.80/6.5 = 10.5 \text{ in.}$$

By equation (12), the approximate modulus of subgrade reaction,

K = (0.0442)(2,000) = 288 lbs. per sq. in. per in.

according to Figure 7, the California Bearing Ratio is approximately equal to 2.5 per cent. Accordingly, the required thickness to carry an 8,000 lb. wheel load is about 22 in.

According to Barber and Palmer (9), the allowable bearing value of the subgrade is:

$$q = \frac{dC_s}{1.5r.}$$

(It should be noted that this formula is slightly different from equation 14)

If d is equal to 0.1,

$$q = \frac{(0.1)\ (2,000)}{(1.5)\ (6.80)} = 19.2 \text{ psi.}$$

Accordingly, the required thickness using Barber and Palmer's formula for the gravel pavement is 10.7 in. and for the asphaltic concrete is 8.4 in.

The required thickness of flexible pavement for q = 10 psi and W = 8,000 lbs. by Down's formula is 15.9 in.; and by the Asphalt Institute formula is 9.2 in. The required thickness using Burmister's solution (2) is 19.4 in. for the gravel pavement and 11.3 in. for asphaltic concrete.

Type of pavement	Required thickness in in. computed according to: Burmister*	Barber	C.B.R.	Down	A.I.	This study
Gravel $E = 10{,}000$ psi.	19.4	10.7	22	15.9	9.2	12.6
Asphaltic concrete $E = 25{,}000$ psi.	11.3	8.4	22	15.9	9.2	10.5

*Computed on the basis that the deflection of surface is equal to 0.1 inch.

(3) Flexural stress in concrete pavements loaded at interior points away from edges or joints.

Given:

Modulus of elasticity of concrete $E_1 = 4{,}000{,}000$ psi.

Wheel load $W = 10{,}000$ lb.

Contact pressure of the tire $p = 70$ psi.

Thickness of the slab $t = 6$ in.

Modulus of elasticity of subgrade soil A, $E_2 = 2{,}000$ psi.

B, $E_2 = 4{,}000$ psi.

C, $E_2 = 8{,}000$ psi.

D, $E_2 = 12{,}000$ psi.

To find the flexural stress in the concrete slabs placed on the four different subgrades as given above.

Solution:

$$r = \sqrt{\frac{10{,}000}{\pi\, 70}} = 6.74 \text{ in.}$$

$$r/t = 6.74/6 = 1.12$$

Subgrade	A	B	C	D
$k = E_1/E_2$	2,000	1,000	500	333
f/p from Fig. 4a or eq. (15)		6.15	4.6	4.0
Flexural stress, f in psi.		429	322	280
Modulus of subgrade reaction by equation (12), K in lb/in.^{3}	66	133	265	398
f in psi by Westergaard's formula	375	347	320	305

(4) Design of interior thickness of concrete pavements (for load stress only).

Given: Data same as in illustrative example (3) except that the allowable stress (f) of the concrete is given as 350 psi.

To determine the required interior thickness of the pavement.

Solution:

$$f/p = 350/70 = 5.0$$

Subgrade	A	B	C	D
$k = E_1/E_2$	2,000	1,000	500	333
r/t from Fig. 4a		0.99	1.19	1.32
Required thickness, t in in.	$7\frac{1}{2}$ appr.	6.7	5.7	5.1

The required interior thickness to carry the given wheel load by Older's formula is 4.6 in. By using a cut and try method, solutions for thickness obtained from Westergaard's formula vary from 5 in. to 3 in.

(5) Design of a flexible pavement based upon the known performance of another pavement of different modulus of elasticity.

Given: Experience has shown that a 10 inch thick macadam pavement with a modulus of elasticity equal to 16,000 psi resting on a subgrade with a modulus of elasticity equal to 4,000 psi has given satisfactory performance under a certain intensity of traffic. For the same traffic and subgrade, what should be the required thickness of a gravel pavement with a modulus of elasticity equal to 8,000 psi? What should be the required thickness of an asphaltic concrete pavement with a modulus of elasticity equal to 25,000 psi? Also, these results are then to be compared with Barber and Palmer's method.

Solution:

Modulus of elasticity of pavement, E_1	8,000	16,000	25,000
Stiffness ratio, $k = E_1/E_2$	2	4	6.3
Relative effectiveness, C from Fig. 8	1.05	1.17	1.30

Required thickness of the gravel pavement, $(10)(1.17)/1.03 = 11.4$ in.

Required thickness of the asphaltic concrete pavement,

$$(10)(1.17)/1.30 = 9.0 \text{ in.}$$

By Barber and Palmer's method:

The required thickness of the gravel pavement,

$$(10)\ \sqrt[3]{16,000/8,000} = 12.6 \text{ in.}$$

The required thickness of the asphaltic concrete pavement

$$(10)\ \sqrt[3]{16,000/25,000} = 8.6 \text{ in.}$$

It should be noted that in Barber and Palmer's method, the modulus of elasticity of the subgrade does not affect the solutions of the type covered in this illustrative example.

(6) Design of a multi-layer flexible airport runway.

Given: Airplane wheel load, $W = 30,000$ lb.

Contact pressure of tire, $p = 70$ psi.

Bituminous top course, $E = 30,000$ psi.

Broken stone base course, $E = 15,000$ psi.

Compacted natural soil, $E = 4,500$ psi. and the allowable bearing value, $q = 25$ psi.

Subgrade soil, $E = 2,000$ psi, and the allowable bearing value, $q = 10$ psi.

Design the thickness for each layer of the pavement.

Solution: The problem of stress distribution in a multilayer system is very complicated. However, an approximate solution can be obtained utilizing the idea of "relative effectiveness" with the results of the single layer system.

The radius of the tire contact area is,

$$r = \sqrt{\frac{W}{\pi p}} = 10.92 \text{ in.}$$

Referring to the subgrade and the subbase course of compacted natural soil, $k = 4,500/2,000 = 2.25$.

$$q/p = 10/80 = 0.125$$

According to Figure 3a, $r/t = 0.35$, and the required thickness of the subbase course for distributing the wheel load is $10.92/0.35 = 31.2$ in.

Referring to the base course and the subbase course,

$$k = 15,000/4,500 = 3.33.$$

$$q/p = 25/80 = 0.313.$$

Again referring to Figure 3a, $r/t = 0.66$, and the required thickness of the base course to distribute the wheel load is $10.92/0.66 = 16.5$ in.

Referring to the subgrade:

for the subbase course, $k = 2.25$, by Figure 8, $C = 1.08$;

for the base course, $k = 15,000/2,000 = 7.5$, $C = 1.37$.

16.5 in. of broken stone course is equivalent to

$(16.5)(1.37)/1.08 = 20.9$ in. of the compacted soil.

The required net thickness of the compacted soil is $31.2 - 20.9 = 10.3$ in.

Referring to the subgrade:

for the top course, $k = 25,000/2,000 = 12.5$, and in Fig. 8, $C = 1.63$;

for the base course, $k = 7.5$, $C = 1.37$.

If the thickness of the top course is 4 in., it will be equivalent to $(4)(1.63)/1.37 = 4.8$ in. of the broken stone course.

The required net thickness of the base course $16.5 - 4.8 = 11.7$ in.

In summary: thickness of the top course, 4in.;

thickness of the broken stone base course, 12 in.;

thickness of the compacted subbase course, 10 in.

SUMMARY AND CONCLUSIONS

1. The analysis provides a method to determine numerically the stresses at any point in a layered system for a circular uniformly distributed load acting on the surface of the layer. The layered system consists of an infinite layer of finite thickness superposing on an adhesive foundation of different modulus of elasticity.

2. The stress analysis developed on the basis of the theory of elasticity reveals the exact effect of the stiffness of the pavement on the distribution of the wheel load by the pavement and on the flexural stress of the pavement. The analysis provides charts which show that:

   a. The maximum subgrade reaction decreases as the modulus of elasticity of the pavement increases.

   b. The distribution of the wheel load on the subgrade becomes wider when the modulus of elasticity of the pavement increases.

   c. The flexural stress in a pavement increases with the modulus of elasticity of the pavement.

   d. The maximum shearing stress at the interface between the pavement and the subgrade decreases as the modulus of elasticity of the pavement increases.

   e. The distribution of the shearing stress at the interface becomes wider and wider as the modulus of elasticity of the pavement increases.

3. The results of the analysis can all be expressed as functions of dimensionless terms. The basic terms involved are: the ratios between the contact pressure of the tire and the stresses investigated, the ratio between the radius of contact area and the thickness of the pavement, the ratio between the modulus of elasticity of the pavement and that of the subgrade, the ratios between the coordinates of the points investigated and the thickness of the pavement. Increasing the first term in each ratio has exactly the same effect as decreasing the second term in the same ratio.
4. The stress analysis furnishes the basis for the "structural design" of highway pavements. To facilitate the practical application of the results, charts for determining stresses and approximate formulas to give reasonably accurate results are presented.
5. If the stiffness ratio is equal to unity, the result of this study check with those obtained by Boussinesq.

For concrete pavement design in general, the values of flexural stress obtained in this study are higher than those obtained by Westergaard's formula. However, for average subgrade conditions, the differences are not appreciable.

For flexible pavement design, this study provides a theoretical proof that suitable concentration factors should be used with current design formulas as advocated by some investigators.

REFERENCES

1. Westergaard, H.M. Stresses in concrete pavements computed by theoretical analysis. Public Roads, Vol.7, p. 25-35. April 1926.

2. Burmister, D.M. The theory of stresses and displacements in layered systems and applications to the design of airport runways. Proceedings, Highway Research Board. Vol. 23, p. 126-148. 1943.

3. Massachusetts Highway Commission. Eighth Annual Report. Jan. 1. 1901.

4. Sheets, F.T. Rational road design. Engineering News Record. Vol. 114, p. 83-89. 1935.

5. Gray, B.E. Design and construction of bituminous pavements. Unpublished report presented at the Annual Meeting of Highway Engineers and Commissioners of Michigan, Houghton, Mich. 1934.

6. Harger, W.Q. and Bonney, E.A. Handbook for Highway Engineers, 4th ed. McGraw-Hill Book Co. 1927.

7. Goldbeck, A.T. A method of designing of non-rigid pavements for highways and airport runways. Proceedings, Highway Research Board, Vol. 20, p. 258-269. 1940.

8. Lelievre, C. and Pons, A. Construction of waterbound macadamized roads. Report No. 81, Proceedings, International Road Congress. 1913.

9. Palmer, L.A. and Barber, E.S. Soil displacement under a circular loaded area. Proceedings, Highway Research Board, Vol. 20, p. 279-286. 1940.

10. Klinger, E.W. A suggested method for analyzing load test data with relation to subgrade support. Proceedings, Association of Asphalt Paving Technologists, p. 223-234. Vol. 15, 1943.

11. Spangler, M.G. The structural design of flexible pavements. Proceedings, Highway Research Board, Vol. 22, p. 199-218. 1942.

12. Housel, W.S. Design of flexible surfaces. Proceedings, Twenty-third Annual Highway Conference, University of Michigan, Vol. 39, p. 113-134. 1937.

13. Vokac, R. Thickness of surface and base courses for flexible pavements. Proceedings, Highway Research Board, Vol. 23, p. 155-161. 1943.

14. Hawthorn, G.E. A method of designing non-rigid highway surfaces. Bulletin No. 83, University of Washington, Engineering Experiment Station, 1935.

15. Office of Chief Engineer, U.S. War Department. Design of runways, aprons and taxiways at air force stations. Engineering Manual, Chapter XX, March 1943.

16. U.S. Civil Aeronautic Administration. Design Manual for Airport Pavements. 1944.

17. Boyd, K. An analysis of wheel load limits related to design. Proceedings, Highway Research Board, Vol. 22, p. 185-195. 1942.

18. Hubbard, P. Flexible pavement reaction under field load bearing tests. Research Series No. 9, the Asphalt Institute. 1943.

19. Glossop, R. and Golder, H.Q. The construction of pavements on clay foundation. Unpublished paper presented before Road Engineering Division, the Institute of Civil Engineers, May 9, 1944.

20. Smith, V.R. Flexible pavement foundation design. Journal of Asphalt Technology, Vol. 3, p. 57-73. No. 3, March-April 1944.

21. Older, C. Highway research in Illinois. Transaction A.S.C.E. p. 218-232. Vol. 87. 1924.

22. Portland Cement Association. Concrete Pavement Design 1946.

23. Timoshenko, S. Theory of Elasticity. McGraw-Hill Book Co. 1934.

24. Kelley, E.F. Application of the result of research to structural design of concrete pavements. Public Roads Vol. 20, p. 83-103. 1939.

25. Spangler, M.G. Stresses in the corner region of concrete pavements. Bulletin 157, Iowa Engineering Experiment Station, Iowa State College, 1942.

26. Bradbury, R.D. Reinforced Concrete Pavements. The Wire Reinforcement Institute, Washington, D.C. 1938.

27. Boussinesq, J. Application des Potentiels a l'Etude de l'Equilibre et du Mouvement des Solides Elastiques. Gauthier Villard, Paris. 1885.

28. Love, A.E.H. The stress produced in a semi-infinite solid by pressure on part of the boundary. Phil. Trans. Royal Society, London. Series A Vol. 228, p. 377-420. 1929.

29. Melan, E. Die Druckverteilung durch eine elastische Schichte. Beton u. Eisen, Vol. 18, p. 87-99. 1919.

30. Marguerre, K. Druckverteilung durch eine elastische Schichte auf starrer rauher Unterlage. Ing. Archiv, Vol. 2, p. 108-117. 1931.

31. Passer, W. Druckverteilung durch eine elastische Schichte. Sitzber. Akad. Wiss. Wien, Abt. IIa, Vol. 144, p. 267-275. 1935.

32. Biot, M.A. Effect of certain discontinuities on the pressure distribution in a loaded soil. Physics, Vol. 6, p. 367-375. 1935.

33. Hogg, A.H. Equilibrium of a thin plate symmetrically loaded resting on an elastic foundation of infinite depth. Phil. Mag. and Journal of Science, London, Edinburgh and Dublin, Vol. 25, p. 354-367. 1937.

34. Murphy, G. Stresses and deflections in loaded rectangular plates on elastic foundations. Bulletin 135, Iowa Engineering Experiment Station, Iowa State College. 1937.

35. Marguerre, K. Spannungsverteilung und Wellenausbreitung in der dicken Platte. Ing. Archiv. Vol. 4, p. 332-353. 1933.

36. The Asphalt Institute. Manual on Hot-mix Asphaltic Concrete Paving. 1943.

37. Middlebrooks, T.A. and Bertram, G.E. Soil test for design of runway pavements. Proceedings, Highway Research Board, Vol. 22, p. 144-173. 1942.

ACKNOWLEDGMENT

The author wishes to express his sincere gratitude to Professors R.A. Moyer and R.E. Gaskell for their helpful suggestions in preparing this thesis.

APPENDIX

To illustrate the detailed procedure for the numerical evaluation of the stresses at any point, $S_{rz}(z = 0, \mu = 0.2)$ is evaluated as follows:

Substitute the values of z and μ into equation 9_a (p. 31):

$$(S_{rz})_1 = (S_{rz})_2 = 3.2\ nap \int_0^{\infty} \frac{J_1(ma)J_1(mr)}{N}\left\{\left[m(2.2 + n) + 0.6(1 - n)\right]e^{m} + (1 - n)(-0.6 - m)e^{-m}\right\} dm$$

$$= 3.2\ nap \int_0^{\infty} J_1(ma)J_1(mr)F(m,n)dm$$

in which $N = -\left\{2.2(1 - n)^2 e^{-2m} + (2.2 + n)(2.2n + 1)e^{2m} - (1 - n)\left[(2.2 + n)(1 + 4m^2) + 2.2(2.2n + 1)\right]\right\}$

The numerical values of N for various values of m and n are as follows:

k = 1/n	m	0	½	1	2	3
1		-10.24	-27.85	-75.7	-560	-4140
10		- .1024	- 1.75	- 8.14	-116.0	-1052
100		- .0010	- .350	- 3.85	- 84.1	- 829
1000		- .00001	- .192	- 3.36	- 80.6	- 806

The numerical values of F(m,n) for various values of m and n are:

k = 1/n	m	0	½	1	2	3
1		0	- .0947	-.1147	- .0846	- .0466
10		0	-1.25	-.884	- .325	- .142
100		0	-6.13	-1.825	- .447	- .179
1000		0	-10.11	-2.085	- .455	- .190

The asymptotic expression F_1 of the function F is obtained by dividing the term of highest order in the numerator of F by the term of highest order in the denominator of the same.

$$F_1 = \frac{-1}{2.2n + 1} \, me^{-m} = Ame^{-m}$$

The numerical values of 1/A for various values of k are:

k = 1/n = 1 , 10 , 100 , 1000.

1/A = -3.2 -1.22 -1.022 -1.0022

Note that: $F(m,1) = -me^{-m}/3.2$.

Therefore $F_1 = 3.2A \, F(m,1)$

The numerical values of F_1 are:

k = 1/n	m : 0	: ½	: 1	: 2	: 3
1	0	-.947	-.1147	-.0846	-.0466
10	0	-.249	-.302	-.222	-.122
100	0	-.296	-.360	-.265	-.146
1000	0	-.303	-.367	-.271	-.149

The numerical values of $F - F_1$ are:

k = 1/n	m : 0	: ½	: 1	: 2	: 3
1	0	0	0	0	0
10	0	-1.00	- .582	- .103	- .020
100	0	-5.83	-1.47	- .182	- .033
1000	0	-9.81	-1.72	- .184	- .040

Let $F - F_1 = m(Be^{-bm} + C\,e^{-cm})$.

Solving for B, C, b, c, we have approximately:

k	B	b	C	c
1	0	0	0	0
10	-6.8	-2.44	-	-
100	-4.45	-2.0	-116	-4.9
1000	-5.25	-2.0	-330	-5.8

Following approximate expressions for F are obtained:

$F(m,1) = -0.312me^{-m}$

$F(m,1/10) = -0.816me^{-m} - 6.8me^{-2.44m}$

$F(m,1/100) = -0.978me^{-m} + 4.45me^{-2m} - 116me^{-4.9m}$

$F(m,1/1000) = -0.998e^{-m} - 5.25e^{-2m} - 330me^{-5.8m}$

With formulas 11a (p. 42), take a = 1:

$$S_{rz}(z = o,\ k = 1,\ a = 1,\ \mu = 0.2)$$
$$= 3.2\ nap \int_0^\infty J_1(ma)J_1(mr)\left(-0.312me^{-m}\right)dm$$
$$= -p\int_0^\infty J_1(m)\ J_1(r)me^{-m}\ dm = -pF_3(1,r)$$

$$S_{rz}\ (z = o,\ k = 10,\ a = 1,\ \mu = 0.2)$$
$$= 0.32\ ap \int_0^\infty J_1(ma)\ J_1(mr)\left(-0.816me^{-m} - 6.8\ me^{-2.44m}\right)dm$$
$$= -p\left(0.262F_3(1,r) + \frac{(0.32)(6.8)}{2.44^2}F_3(0.41,\ 0.41r)\right)$$
$$= -p\left(0.262F_3(1,r) + 0.365F_3(0.41,\ 0.41r)\right)$$

Also:

$$S_{rz}\ (z = o,\ k = 100,\ a = 1,\ \mu = 0.2)$$
$$= -p\left\{.0313F_3(1,r) + .0374F_3(0.5,\ 0.5r)\right.$$
$$\left. + .1574F_3(.204,.204r)\right\}$$

S_{rz} ($z = 0$, $k = 1000$, $a = 1$, $\mu = 0.2$)

$= -p\left[.00319F_3(1,r) + .00403F_3(0.5, 0.5r) + .0314F(.173, .173r)\right].$

By formulas 11 (p. 42):

$$F_3(a,r) = \frac{1}{2\pi^2}\int_0^{\pi} d\theta \int_0^{2\pi} d\phi \frac{\sin\theta \sin\phi\left[1-(a\sin\theta + r\sin\phi)^2\right]}{\left[1 + (a\sin\theta + r\sin\phi)^2\right]^2}$$

Because of the symmetry of the integrand

$$F_3(a,r) = \frac{2}{\pi^2}\int_0^{\pi/2} d\theta \int_{-\pi/2}^{\pi/2} d\phi \frac{\sin\theta\sin\phi\left[1-(a\sin\theta + r\sin\phi)^2\right]}{\left[1 + (a\sin\theta + r\sin\phi)^2\right]^2}$$

Let $I(\theta,\phi) = \dfrac{\sin\theta\sin\phi\left[1 - (a\sin\theta + r\sin\phi)^2\right]}{\left[1 + (a\sin\theta + r\sin\phi)^2\right]^2}$

To find F_3, the numerical values of I for various values of θ and ϕ must be found. For example, in $F_3(0.5,1)$ the values of I are:

θ° \ ϕ°	-90	-60	-30	0	30	60	90
0	0	0	0	0	0	0	0
30	-.090	-.141	-.207	0	+.090	-.023	-.043
60	-.230	-.431	-.427	0	+.032	-.071	-.089
90	-.480	-.583	-.500	0	0	-.091	-.120

Now we apply the trapezoid rule twice to those tabulated values to evaluate the double integral:

$$F_3(0.5,1) = \frac{2}{\pi^2}\left(\frac{\pi}{6}\right)^2\left(\frac{1}{4}\right)\sum_{i=1}^{.72} I_i = \frac{1}{72}\sum_{i=1}^{72} I_i$$

$$= \frac{-1}{72}\Big[(0 + 2(0) + 2(0) + 2(0) + 2(0) + 2(0) + 0 + 2(.090)$$

$$+ 4(.141) + 4(.207) + 4(0) + 4(.090) + 4(.023) + 2(.043)$$

$$+ 2(.230) + 4(.431) + 4(.427) + 4(0) + 4(.032) + 4(.071) + 2(.089)$$

$$+ .480 + 2(.583) + 2(.500) + 2(0) + 2(0) + 2(.091) + .120\Big]$$

$$= -8.808 \div 72 = -0.1225.$$

Similarly, numerical values of the function for other arguments can be evaluated. Finally the numerical value of the stress at any point can be obtained. For example:

$r =$	0.	1.	2.	3
$F_3(1,r) =$	0	-.182	-.0662	-.0121
$F_3(0.41,\ 0.41r) =$	0	-.136	-.129	-.0699
$-0.262F_3(1,r) =$	0	.0476	.0173	.0029
$-0.366F_3(0.41,\ 0.41r) =$	0	.0496	.0470	.0256
$\frac{Srz}{p}$ ($z = o$, $k = 10$, $a = 1$ $\mu = 0.2$) $= -.0262F_3(1,r) - 0.366F_3(0.41,\ 0.41r) =$	0	.0972,	.0643	.0285.

(See Figure 5, p. 46).